Causal Loops

Logically Consistent Correlations, Time Travel, and Computation

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I certify that except where due acknowledgement has been given, the work presented in this thesis is that of the author alone; the work has not been submitted previously, in whole or in part, to qualify for any other academic award; and the content of the thesis is the result of work which has been carried out since the official commencement date of the approved research program.

Ämin Baumeler
Lugano, 30 March 2017

To this dedication

To my parents, Hanspeter and Zohra

In some remote corner of the sprawling universe, twinkling among the countless solar systems, there was once a star on which some clever animals invented knowledge. It was the most arrogant, most mendacious minute in world history, but it was only a minute. After nature caught its breath a little, the star froze, and the clever animals had to die. And it was time, too: for although they boasted of how much they had come to know, in the end they realized they had gotten it all wrong. They died and in dying cursed truth. Such was the species of doubting animal that had invented knowledge.

Peacerich Nothing

In irgendeinem abgelegenen Winkel des in zahllosen Sonnensystemen flimmernd ausgegossenen Weltalls gab es einmal ein Gestirn, auf dem kluge Tiere das Erkennen erfanden. Es war die hochmütigste und verlogenste Minute der 'Weltgeschichte'; aber doch nur eine Minute. Nach wenigen Atemzügen der Natur erstarrte das Gestirn, und die klugen Tiere mußten sterben. Es war auch an der Zeit: denn ob sie schon viel erkannt zu haben sich brüsteten, waren sie doch zu letzt, zu großer Verdrossenheit, dahinter gekommen, daß sie alles falsch erkannt hatten. Sie starben und fluchten im Sterben der Wahrheit. Das war die Art dieser verzweifelten Tiere, die das Erkennen erfunden hatten.

Friedrich Nietzsche

Abstract

Causal loops are loops in cause-effect chains: An effect can be the cause of that effect's cause. We show that causal loops can be unproblematic, and explore them from different points of view.

This thesis is motivated by quantum theory, general relativity, and quantum gravity. By accepting all of quantum theory one can ask whether the possibility to take superpositions extends to causal structures. Then again, quantum theory comes with conceptual problems: Can we overcome these problems by dropping causality? General relativity is consistent with space-time geometries that allow for time-travel: What happens to systems traveling along closed time-like curves, are there reasons to rule out the existence of closed time-like curves in nature? Finally, a candidate for a theory of quantum gravity is quantum theory with a different, relaxed space-time geometry.

Motivated by these questions, we explore the classical world of the non-causal. This world is non-empty; and what can happen in such a world is sometimes *weird*, but not *too crazy*. What is weird is that in these worlds, a party (or event) can be in the future *and* in the past of some other party (time travel). What is not too crazy is that this theoretical possibility does not lead to any contradiction. Moreover, one can identify *logical consistency* with the existence of a *unique* fixed point in a cause-effect chain. This can be understood as follows: No fixed point is the same as having a contradiction (too stiff), multiple fixed points, then again, is the same as having an unspecified system (too loose).

This leads to a series of results in that field: Characterization of classical non-causal correlations, closed time-like curves that do not restrict the actions of experimenters, and a self-referential model of computation. We study the computational power of this model and use it to upper bound the computational power of closed time-like curves. Time travel has ever since been term *weird*, what we show here, however, is that time travel is not *too crazy*: It is not possible to solve *hard* problems by traveling through time.

Finally, we apply our results on causal loops to other fields: an analysis with Kolmogorov complexity, local and classical simulation of PR-box correlations with closed time-like curves, and a short note on self-referentiality in language.

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Contents

Chapter 1

Introduction

> Wenn [...] Raum, Zeit, und Causalität als gänzlich unbedingte Gesetze von allgemeinster Gültigkeit behandelt [werden, dann ist] die blosse Erscheinung, das Werk der Maja, zur einzigen und höchsten Realität zu erheben und sie an die Stelle des innersten und wahren Wesens der Dinge zu setzen und die wirkliche Erkenntnis von diesem dadurch unmöglich zu machen, *d.h.*, nach einem Schopenhauer'schen Ausspruche, den Träumer noch fester einzuschläfern.[1]
>
> Friedrich Nietzsche [185]

A causal structure describes (possible) *cause-effect* relations between events. In relativity theory, for instance, an event is said to be in the *causal future* of another if it resides in the future light cone of the latter. By that, the latter — the cause — can influence the former — the effect. In circuit models of computation, then again, causal structures are given by the configurations of the gates and wires. To have an intuitive picture of causal structures, one can imagine them as directed graphs, where the vertexes are events (or computer instructions), and where every edge points from a *cause* to an *effect* (see Figure 1.1 and Figure 1.2).

Causal structures are usually assumed to reflect *definite partial orderings* of events. For a causal structure to be *definite* means that the causal relations — the configurations

[1] If space, time, and causality are treated as totally absolute laws with the most universal validity, then this serves only to raise the mere appearance, the work of Maja, to the single, highest reality and to set it in place of the innermost and true essence of things and thus to render actual perception of this essence impossible, *i.e.*, in the words of Schopenhauer, to put the dreamer asleep even more deeply.

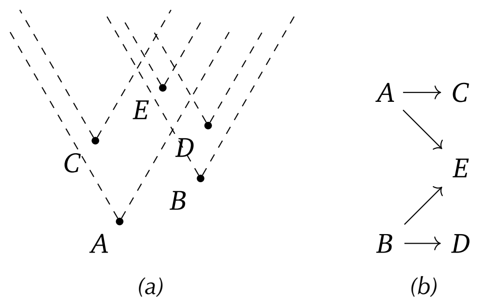


Figure 1.1. (a) Events in space-time with respective light cones. (b) Causal structure from the setup in (a).

— are *predefined*. This does not rule out probabilistic relations, but asserts that the relations are independent of any observer. Furthermore, that causal relations describe a *partial ordering* implies that they do not contain *causal loops*, *cycles*: *An effect cannot be a cause of the effect's cause*. These assumptions seem to be built into our intuition (one could say we have been conditioned to that point of view), and guide use in how we perceive the world and fantasize.

In this thesis, we trespass any such perception or fantasy and imagine the *non-causal*: *indefinite* causal relations, *causal loops*, or combinations of both. Although we take the reader to that world, this thesis is best read causally: Sentence after the sentence.[2] The following part is of particular relevance since it *motivates* this expedition on *physical* grounds and presents the possible implications for computer science.

1.1 Motivation and consequences

The work performed in this thesis is motivated by *quantum theory*, *relativity theory*, and *quantum gravity*. We successively comment on each of the motivations. A detailed account of the motivations is found in the next chapter.

- Since ever **quantum theory** has been formulated, debates [63] on how to interpret the theory are lasting: How should one interpret the wavefunction, what is a measurement, where does a measurement happen, *etc.*? So, quantum theory confronts us with a series of foundational questions that many people tried to solve. Possibly these barriers can be overcome by waiving the Newtonian notion of *absolute* space and time, and by approaching the problems from a Leibnizian [76] view

[2]As opposed to the *magnum opus* of Schopenhauer [223] (which is kind of non-causal), in which the author tells to the reader: "Es ergibt sich von selbst, daß, unter solchen Umständen, [. . .] kein anderer Rath ist, als *das Buch zwei Mal zu lesen* [. . .]." — "It is self-evident that under such circumstances no other advice can be given than to read the book twice." Schopenhauer backs this recommendation by saying that obviously the end of his book depends on the beginning, but so does the beginning on the end.

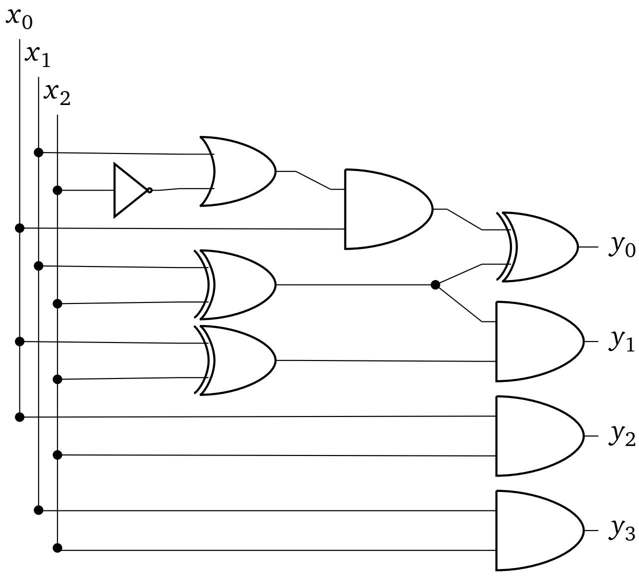


Figure 1.2. A redrawing of a circuit from Reference [162]. It is clear that such a circuits admits a *definite partial ordering* of the applications of the gates.

instead: Space and time are subject to the physical laws as well. Furthermore, so-called *Bell [40] non-local* correlations seem to avoid our classical intuition on how nature should behave, and forces many to attribute some "spookiness" to the theory. However, it cannot be stressed enough that this latter inconvenience might be a consequence of some (hidden) assumptions: *e.g.*, the specific causal structure used to describe the correlations. Quantum theory motivates this research even more by allowing physical quantities to be in superpositions or entangled with others. So, it seems natural to allow causal structures to be in superpositions too [15, 16, 52, 68, 114, 167, 173, 191, 192, 194, 201] (which has been verified experimentally [207, 217]).

- **General relativity** motivates this study because some geometries thereof contain *closed time-like curves* [131, 165] — causal loops. A system traveling on such a curve potentially meets its younger self and interacts with it. Questions that arise in that context are the following two. Are such geometries simply a mathematical artefact? What are the dynamics such systems undergo?

- Finally, if we want to combine quantum theory with general relativity — to form some theory of **quantum gravity** — then it seems reasonable [146] to drop some assumptions on causal structures: Relativity theory is a *deterministic* theory with a *dynamic* space-time, and opposed to that, quantum theory is *probabilistic* (at least as claimed by many) with a *fixed* causal structure. So, if both theories would be merged, then the resulting theory is speculated to contain the *less restrictive* features of both "mother" theories. The interplay between quantum theory and relativity motivates this study even further: Are *Bell non-local correlations* simulatable in geometries that contain closed time-like curves?

These motivations can be put in two categories: (1) the idea of replacing the "spookiness" of quantum theory with weaker assumptions on causal structures, and (2) the idea of accepting quantum theory as it is to apply it to causal structures, *e.g.*, to allow causal structures to be in superposition.

Studies in this field might have wide-ranging *consequences*. Following the motivations, such studies might (1) allow us to have a classical understanding of the quantum phenomena, or (2) force us to change our view on the notion of causality. New views on causality, then again, have radical impact on other concepts like *time*. Furthermore, insights might facilitate to combine quantum with relativity theory (*e.g.*, References [64, 114, 256]). In computer science, potential consequences of such studies are improved information-processing capabilities, *e.g.*, faster computation. The role of computer science should not be underestimated: In this thesis, computer science approaches proved themselves useful to make statements on the *physical nature* of closed time-like curves.

1.2 Antinomies

Dropping the assumption that causal structures are *definite* and do not contain *cycles* comes with a caveat: *logical inconsistencies*. In a causal loop, for instance, nothing seems to forbid an *effect to suppress the cause of the effect's cause*, leading to the problem best known as the *grandfather antinomy*.[3] Suppose someone travels to the past and kills his or her grandfather before that time traveler was conceived. By doing so, he or she cannot be born, and by that cannot travel to the past to kill the grandfather. But since the grandfather was not killed, the person is born and can travel to the past to kill… — a *logical contradiction*.

Another antinomy in the same spirit (it can be understood as the other extreme) is the *information antinomy*. Suppose one morning someone wakes up to find a book on the table that he or she never has seen before. This person then publishes this book (which is irrelevant for the antinomy), and travels to the past to place the book on the table in the night before the very same person has found the book. Here, no logical contradiction in the above sense seems to arise. However, this *causal loop* has another problem: The content of the book is *unspecified, i.e.*, it arises *ex nihilo*. This means, any such causal loop leaves open what *states* the physical systems should be in — *multiple consistent* solutions exists. Deutsch [93] sees this antinomy as more *severe*

[3]This antinomy is often called "grandfather paradox," which however is not accurate. The noun "paradox" originates from the Greek word *paradoxon*, which is composed out of *para* (against) and *doxa* opinion. So, a "paradox" is a *seeming*, as opposed to an *actual*, contradiction. Instead, we use the term *antinomy* to refer to logical problems.

than the grandfather antinomy because it undermines any scientific theory, and follows the doctrine of *creationism*. The states on the loops spring fully formed into the universe without emerging from some process (every consistent solution is equally justified).

We hope to make these antinomies clearer by giving two analogies: A definition of the form "*x is defined as not x*," (contradiction) is of the first kind, and "*x is defined as x*" (tautology) is of the second kind.

In this thesis, we circumvent these antinomies by the assumption of *logical consistency*, which depends on the model considered. In one model, *logical consistency* is the assumption that *probabilities*, under any choice of operations of some parties, are *well-defined*. In another model, we replace *probabilities* with *dynamics*. In the computational model, then again, these two antinomies are explicitly avoided. One can understand the works performed in this thesis as an expedition to the world between the *logically consistent* and the *causal* — similar in spirit to the studies of Bell non-local correlations that are non-signaling.

1.3 Main results and outline

We study relaxed causal structures in different contexts: *Correlations, closed time-like curves, and computation*.

- A **classical**[4] **model for correlations** is designed, where local assumptions and the assumption of logical consistency only are made (see Chapter 4). We show that this model allows for *non-causal* correlations and *causal loops*. This model is proven to be the *classical special case* of the *process-matrix framework* — a quantum model with local assumptions only. We give different characterizations of the allowed processes in the model, which lead to a better understanding of *logical consistency*. Then, we show that all processes in that model can be made *reversible*. By that, the model is not only *non-causal* but also *reversibly non-causal*. This last result allows us to design a *unitary* (reversible) quantum operation that violates the causality assumptions as well. Thus, *either* not every reversible dynamics is physically realizable, *or* causal loops are. Some of these results appear in some of our articles (see References [28, 31, 32, 34]).

- The dynamics of **closed time-like curves** have been partially studied in the context of general relativity. Researches around Thorne [97] analytically throw a billiard ball into a closed time-like curve to let it interact with itself. Their studies, however, are limited to the case where at some point, the initial conditions are set, and afterwords, the billiard ball is left to the dynamics. Interestingly, for all initial conditions studied, an *infinity* of consistent dynamics were found where

[4]Throughout this thesis a "classical model" is understood as a model where the underlying states are constructed out of bits or out of probability distributions over bits.

the billiard ball self-interacts. A drawback of these studies is that a potential experimenter interfering with the time-traveling billiard ball is disregarded. The actions of such an experimenter, however, might forbid any consistent dynamics. We show (see Chapter 5) that this is not the case by explicitly designing a closed time-like curve where some experimenters in closed regions are allowed to interact arbitrarily — in any way they wish — with the time traveling systems. These dynamics, just as in the first part, can be made reversible. Finally, we compare this model to other models to predict dynamics of closed time-like curves. Some of these results appear in some of our articles (see Reference [36, 37]).

- In the last main part of this thesis, we describe a **circuit model of computation** that avoids the grandfather and the information antinomies. This model is shown to be more powerful in terms of *query complexity* when compared to classical circuits. Then, we study the *computational power of that model* and find it to be characterized by the complexity class $\mathsf{P} \subseteq \mathsf{UP} \cap \mathsf{coUP} \subseteq \mathsf{NP}$. A known problem within that complexity class is *integer factorization*. We use this problem as a case study and design a circuit that factorizes integers *efficiently* within that model. This result has implications fo the middle part of the thesis: The computational power of the closed time-like curves is *upper bounded* by $\mathsf{UP} \cap \mathsf{coUP}$. Thus, in *stark contrast* to other models of closed time-like curves [6, 48], this model *cannot* solve NP-complete problems efficiently (unless $\mathsf{UP} \cap \mathsf{coUP} = \mathsf{NP}$, which is highly doubted). Some of these results appear in some of our articles (see References [30, 33]).

The final chapter contains some applications of the results to other fields: Kolmogorov complexity (see Reference [29]), PR-box correlations from classical closed time-like curves (see Reference [38]), and self-referentiality in language.

In the next chapter, we give a detailed account of the motivations for the works performed here. After that, the main part of the thesis begins with a forgoing chapter on *causality* followed by the three *building blocks* of the thesis: *Correlations, time travel, and computation*. We approach the end with a chapter where results are applied to different fields and with more speculative content. Finally, we list some open questions and conclude the thesis. The appendix consists of a short remark on the notation and on preliminaries, which serve as a reference. This thesis comes with an epigraph — a short and personal notice on *anti-realism*. It does not contain technical results, but puts them in a philosophical and sociological context.

Chapter 2

Motivations

In dieser Weise erinnert uns [...] die Quantentheorie daran, daß man beim Suchen nach der Harmonie im Leben niemals vergessen darf, daß wir im Schauspiel des Lebens gleichzeitig Zuschauer und Mitspielende sind.[1]

Werner Heisenberg [151]

Newton advocated a view where space and time are *absolute*. By that, he considered "space and time as a 'scene' upon which the drama of the evolution of the [u]niverse was taking place" [190]. Newton's view can be contrasted by Leibniz's: There is no absolute space and time, but space and time are formed through *relations*. Corresponding to the analogy above: Space and time emerge with, and are part of, the drama. This dichotomy led to a long debate between Leibniz [76] and Clarke — a supporter of Newton.[2] Nowadays, it seems as if Newton won that debate: Relativity as well as quantum theory presuppose some form of preexisting space-time.

Both of general relativity and quantum theory — as good as they are — come with peculiarities that timidly call after Leibniz. In the next two sections, we present some of these peculiarities, and use them as motivations for this thesis. Furthermore, ever since these two theories have been developed, it remained a quest to merge them to a unified theory of *quantum gravity*. That quest motivates the works presented here as well; details on that are given in the third section of this chapter.

[1] In this way, quantum theory reminds us that in the search of the harmony in live one may never forget that we are simultaneously a spectator and performer of the play of life.

[2] It is believed that Newton contributed substantially to Clarke's replies (see, *e.g.*, References [166, 240]).

2.1 Quantum theory: EPR correlations

Καὶ τὸ ὅλον τοῦ μέρους μεῖζόν ἐστιν.[3]

Euclid of Alexandria [110]

Quantum theory allows for correlations among parties that shatter our classical perception of how nature should behave: Physical quantities seem to arise from *nowhere* and yet, distant quantities are *correlated*.[4] Already Einstein [100, 101] was doubtful about the formulation of quantum theory because of that issue. Let us elaborate. Within quantum theory, it is possible to design quantum systems that cannot be described by the sum of their parts. By this we mean that the attempt to describe the whole system by describing its subsystems only fails. This, however, at most questions the *mathematical formalism*. A seminal critique of the formalism — based on this *inseparability* — was undertaken by the trio Einstein, Podolsky, and Rosen [107] (EPR). This critique is reproduced in the following section. Their concerns led to a series of works on the issue and eventually to experimental demonstration of so-called Bell [40] non-local correlations. Drastic implications of these correlations seem unavoidable as long as we hold to the *classical* world-view. But are these implications avoidable at the expense of allowing *causal loops* in physical theories?

After the reproduction of the EPR critique, we discuss the underlying assumptions that the trio made. Then, we present the implications, and some ways out: *Relative*, *retro*, and *emergent* causality.

2.1.1 The Einstein-Podolsky-Rosen argument

Einstein, Podolsky, and Rosen conclude that the quantum-mechanical *description of reality is incomplete*. They do so by holding on to a certain notion of *reality* and based on that, question the theory. To understand their argument, we first need to clarify the terms *reality*, *description of reality*, and *completeness of a theory*.

- The notion of *reality*, in a basic form, asserts the existence of entities *independent* of the observer (see, *e.g.*, References [25, 96]). In other words, *realism* is the concept that nature's *answers* to questions we pose by performing experiments *exist prior* to having performed the experiments. Einstein, Podolsky, and Rosen do not give a full account of reality, but give a *sufficient condition* for what counts as *element of reality*: "If, without in any way disturbing a system, we can predict with certainty (*i.e.*, with probability equal to unity) the value of a physical quantity,

[3] And the whole is greater than the part.

[4] Note that some would disagree with this first statement that correlations arise from nowhere; they would counter by saying that the correlations do come from the *quantum states being measured*. In that case, we ask the reader to bare with us and we repeat: They *seem* to arise from nowhere and are correlated.

then there exists an element of physical reality corresponding to this physical quantity." Note that this form of *reality* is independent of any physical theory.

- *Description of reality* is an expression from *philosophy of science* and it means that *some elements of reality* have their corresponding elements in a physical theory. Crucially, this notion is bound to a physical theory.

- Finally, for a theory to be *complete*, "every element of the physical reality must have a counterpart in the physical theory" [107].

The argument with which they show that the quantum-mechanical description of reality is *not complete* is the following. First, consider a single physical system S and two physical quantities corresponding to the observables P and Q that do *not commute e.g.*, the observables corresponding to the position and to the momentum of S. If the physical quantity corresponding to P of S is known, then the quantum-mechanical description of the state of S does not contain any information about the quantity corresponding to Q, and *vice versa*. This is known as the Heisenberg [150] *uncertainty principle*. By this first consideration, either the quantum-mechanical description of the state of S is *incomplete* or both quantities are *not simultaneously real*.

Towards a contradiction, assume that the quantum-mechanical description *is complete*, and consider two physical systems, S_1 and S_2, which initially interact in way that the joint state becomes *entangled*. After that interaction, suppose that both systems become space-like separated by sending them in opposite directions. Since both systems are in a specific entangled state, if we now measure system S_1 with some observable P, then we can predict with *certainty* the physical quantity corresponding to P of the other system. The same holds if we measure system S_2 with the observable Q; then, the physical quantity corresponding to Q of S_2 can be predicted with certainty. At this point, it is crucial that both systems are *space-like* separated. This means that a measurement on S_1 cannot influence S_2, *i.e.*, system S_2 remains *unchanged*. By this, since S_2 did not undergo any change and since both quantities can be predicted perfectly, both correspond to *elements of reality*. But that would mean that the theory is *complete* (by assumption) and that both quantities have *simultaneous reality*, as just demonstrated. This statement, however, is in contradiction with the considerations of the single system S above. The resort at first sight is to consider the quantum-mechanical description of reality as *incomplete*. By that, Einstein, Podolsky, and Rosen "have shown that the wave function does not provide a complete description of the physical reality," and they "left open the question of whether or not such a description exists" [107]. Other presentations of the argument are given by Bohm [49] (see also Reference [50]) and Redhead [210].

2.1.2 Local realism and Reichenbach's principle

The assumptions of Einstein, Podolsky, and Rosen are sometimes summarized by the term *local realism*.

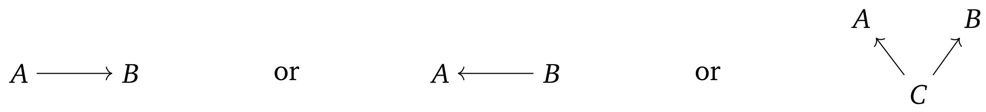


Figure 2.1. Reichenbach's principle: If two physical quantities (A, B) are correlated, then either one is the cause of the other, or both have a common cause C. The arrows describe *cause-effect* relations and point to the effect.

Definition 1 (Local realism, as formulated in Reference [58])**.** *Local realism* is the assumption that results of the observations on the individual systems are predetermined and independent of whatever measurements might be performed distantly.

Local realism can be seen as a combination of *realism* and *locality*. The former asks for the results of some experiment to be *predetermined*. That is, the physical quantities exist *prior* to the experiment as opposed to *emerging through the observation*. Thus, it should not be the case that "[d]ie 'Bahn' [. . .] erst dadurch [entsteht], daß wir sie beobachten"[5] [150]. The latter assumption of *locality* asserts that an object is not influenced by manipulations on space-like separated objects. This assumption is motivated by Einstein's theory of relativity, where the speed of influences cannot exceed the constant speed of light.

A *local-realistic theory* is a theory obeying the above definition. One might wonder how such a theory functions and how correlations within such a theory come about. At this point, we bring in the seemingly *natural* principle of Reichenbach [212], what might be considered as at the heart of *understanding correlations*.

Reichenbach's principle. If two quantities are *correlated*, then one quantity *directly influences* the other, or both *have a common cause* (see Figure 2.1).

This principle, however, is substantially debated; especially since the so-called *Bell non-local correlations* have become undeniable (see the following section). To comply with *Bell non-local correlations*, Reichenbach's principle must either be dropped or modified. An attempt of a *quantum* version of Reichenbach's principle is undertaken by Allen *et al.* [12]. Note that Reichenbach formulated his principle in the context of *macrostatistics* — measurement settings and outcomes are clearly within that realm, this, however, is not necessarily the case for the *quantum systems per se*.

With the formulation of *local hidden-variable models*, Bell aimed at describing a model as general as possible that is *local realistic*. In the following two sections, we show that *Bell non-local correlations* do not admit a *common cause* in such a model and that *direct causation* can be ruled out as well — even if we allow for *superluminal* causation.

[5] "the 'path' comes into existence through our observation."

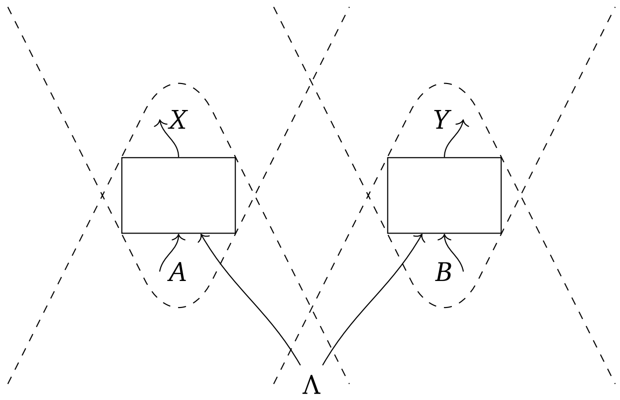


Figure 2.2. Alice, on the left, chooses some setting A and obtains the measurement result X. Bob, on the right, adjusts his measurement apparatus according to the setting B and measures Y. The boxes abstractly represent the measurements. The dashed lines represent light cones. Time flows from bottom to top.

2.1.3 No common cause

Around 30 years after the EPR paper was published, Bell [40] tried to answer the question raised by the trio (see above)[6]: Does there exist a *complete local-realistic description of reality*? In a series of papers [44], Bell aimed to show that a local-realistic model for the quantum-mechanical predictions does not exist. These results, however, are subject to some additional, in parts implicit, assumptions (we elaborate more on those in later parts of this chapter).

Bell's argument is the following. Consider two parties, Alice and Bob, who are space-like separated. Every party freely chooses a measurement setting and observes the measurement result when some system is measured according to that setting. Their choice is *free* in the sense "that the values of such variables have implications only in their future light cones" [43]. Through the space-like separation of Alice and Bob, we ensure that both parties *cannot communicate*. Let us denote by A the random variable that corresponds to Alice's choice of the measurement setting, and B likewise for Bob's choice. We label the outcomes by the random variables X and Y, where X is Alice's and Y Bob's outcome (see Figure 2.2). By that, the whole setup — without referring to some underlying systems — is described by some conditional probability distribution $P_{X,Y|A,B}$. The puzzling feature of quantum correlations is unveiled by studying the probability distributions of this form: Does there exist some *underlying local-realistic model* such that the resulting distribution $P_{X,Y|A,B}$ equals the predictions of quantum theory?

Towards answering this question, Bell developed a so-called *local hidden-variable*

[6]An overview of other attempts to resolve the paradox pointed to by Einstein, Podolsky, and Rosen is given by Bell [41] himself.

model. Such a model, as is shown later, is *local realistic*. Yet, it is just *one* local-realistic model. If by the term *"underlying local-realistic mechanism"* we understand a *local hidden-variable model*, then the question is answered *negatively*: There does *not* exist a model that can give the same predictions as quantum theory.

The underlying system of a *local hidden-variable model* is a *hidden random variable* Λ. This random variable is not constrained in its dimension nor distribution. The only assumption with respect to Λ is that it is a *random variable*. By this we ensure that the system is *realistic* — the state of Λ is *predetermined*. Furthermore, Λ can be understood as being generated in the common past of Alice and Bob (see Figure 2.2). After that system Λ has been distributed to Alice and Bob, each party performs a measurement with his or her corresponding setting, and obtains the measurement result. By the *locality* assumption, the measurement result of one party is not allowed to depend on the setting or result of the other. Rather, it is completely specified by the party's setting and Λ. By putting everything together, a distribution $P_{X,Y|A,B}$ resulting from a *local hidden-variable model* is called *Bell local*, and can be decomposed as in Definition 2.

Definition 2 (Bell locality)**.** A probability distribution $P_{X,Y|A,B}$ is called *Bell local* if and only if it can be expressed as

$$P_{X,Y|A,B} = \sum_{\lambda} P_{\Lambda}(\lambda) P_{X|A,\Lambda=\lambda} P_{Y|B,\Lambda=\lambda} \,.$$

The *freeness* assumption that we mentioned above is incorporated with the independence of Λ from the settings A and B. Finally, the *realism* feature of such a model, as already stated, is given by assuming that the underlying state Λ is a random variable. There exists a vast amount of literature on the derivation of Bell locality, even by Bell [44] himself (see also the article by Brown and Timpson [56] and the blog entry by [13]).

Bell [40] noticed, that a *local hidden-variable model* can be constructed that gives the *same predictions as quantum mechanics* in the Einstein, Podolsky, and Rosen [107] experiment (see Section 2.1.1), when both space-like separated physical systems are measured in the *same* basis: One can simply regard Λ as shared information that determines how these systems "reply" to the *same* measurements. In that sense, one is tempted to conclude that the quantum-mechanical description of reality is *incomplete*, with the local hidden-variable model being the *complete* description. However, this is not the case anymore if we consider *different* measurements on the space-like separated systems.

To show that quantum correlations cannot be simulated by *local hidden-variable models*, Bell [40] arranged the probabilities from some distribution $P_{X,Y|A,B}$ in a way to obtain an *inequality* that is *satisfied* by every *Bell-local* distribution. Such an inequality is called *Bell inequality*. Read in the contrapositive: Whenever a distribution $P_{X,Y|A,B}$ *violates a Bell inequality*, then it is *Bell non-local*, *i.e.*, the distribution cannot be derived from a local hidden-variable model. An example of a Bell inequality in the two-party scenario with binary settings and binary outcomes is (see also Reference [77])

$$\Pr(X \oplus Y = AB) \leq \frac{3}{4},$$

where $\oplus$ is the sum modulo 2, and the settings are assumed to be uniformly distributed. As it turns out, quantum theory *violates* this inequality and hence allows for *Bell non-local* correlations. This has been verified experimentally (see References [19, 20, 121, 128, 129, 153, 220, 222, 224, 235, 245] for a selection of such experiments).

Magic-square game

The above means that one cannot find a *common cause* in the local hidden-variable model that yields the quantum-mechanical predictions. To illustrate how puzzling that is, we introduce the *magic-square game* [179, 200]. In the following scenario, two parties, Alice and Bob, collaboratively play a game — they do not play against each other but aim at winning the game *together*. Before the game starts, the parties are allowed to meet and discuss their strategy. Thereafter, both parties are separated and not allowed to talk to each other anymore. The game they play is the following. Every party is given a 3×3 grid. Alice randomly chooses a *row* $i \in \{1, 2, 3\}$ and fills the row i of her grid with -1s and $+1$s. Analogously and independently, Bob randomly chooses a *column* $j \in \{1, 2, 3\}$ and fills the column j of his grid with $-1s$ and $+1$s. The conditions for both parties to win are

- Alice's value on the jth column has to match Bob's value on the ith row (the common entries match),
- the product of Alice's entries is -1, and
- the product of Bob's entries is $+1$.

This is repeated an arbitrary number of times. Since neither Alice nor Bob knows what the other party's choice is, they must agree on all 9 entries beforehand, *i.e.*, when they discuss the strategy. However, no 3×3 grid with predetermined values exist that satisfies the conditions stated above (see Figure 2.3). So, eventually, the parties *lose* the game. In contrast to this, if Alice and Bob were allowed to share *quantum* states, then in every round Alice and Bob would be able to fill in a row or a column, respectively, with entries that would satisfy the conditions posed; *i.e.*, they always *win* the game. In other words, when the parties have access to quantum states, it seems as if they would communicate — but they do not.

On top of that, Bell non-local correlations lead to a series of consequences, *e.g.*, it leads to an *all-or-nothing* feature of randomness: If the experimenters measuring the quantum states can *freely* choose the settings, then it follows that the physical systems must *freely* "choose" the outcomes as well [82, 83]. In relation to that, Bell non-local correlations also lead to an *all-or-nothing* flavor to the Church-Turing hypothesis:

Bob

Alice

+1	+1	−1
+1	−1	+1
+1	−1	?

Figure 2.3. Alice chooses a random row and Bob chooses a random column to fill in with entries ±1. However, no 3×3 grid of predetermined values exist such the requested conditions are satisfied.

"beyond-Turing computations are either physically impossible, or they can be carried out by 'devices' as simple as photons" [251].

2.1.4 No direct causation

In the previous section, we showed that quantum correlations cannot be modeled with a *common cause* in a *local* hidden-variable model. One might ask: Could there be some *hidden influence* (as opposed to a *common cause*) that leads to the quantum-mechanical predictions? Einstein's theory of relativity rules out any such hidden influence that travels at the speed of light or slower.

We discuss the implications of such hidden influences. The discussion is split in two parts: the *quantitative* nature of hidden influences — giving *promising* results for explaining Bell non-local correlations via hidden influences —, and the *qualitative* nature of hidden influences — *criticizing* hidden influences.

Quantitative nature of hidden influences

Let us take a look on *how much* hidden communication is required to classically simulate any quantum statistics. The scenario is the following: Alice and Bob share some quantum system (possibly entangled) that is measured by each party separately. This scenario is now simulated in a model with hidden influences. For that, we can think of Alice first holding both parts of the quantum system, and measuring her share only. Thereafter, Alice sends *classical* information, *i.e.*, a bit string, to Bob. Bob now *simulates* the measurement on his part of the quantum system by having access to that bit string and some shred classical information (see Figure 2.4). This question of the *communication complexity* of simulating quantum correlations was raised, among others, by Maudlin [176]. He figured out that, in the average, a *finite* amount of bits sent from Alice to Bob is enough to simulate a quantum measurement. A finite amount of classical information seems very surprising, as quantum states and measurements are

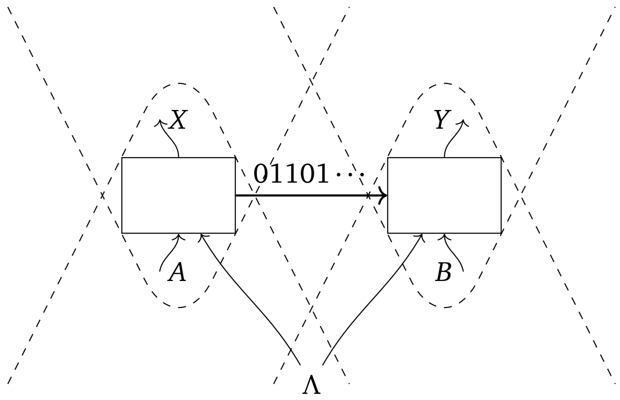


Figure 2.4. Alice, on the left side, measures her share with the setting A. Bob, on the right, *simulates* the measurement on his share. The simulation is based on the classical information obtained from Alice, on some shared random variable Λ, and on his setting B. The outcome of the simulated measurement is Y.

characterized by real values. Maudlin's result was strengthened by Brassard, Cleve, and Tapp [55]: A finite amount of communication is sufficient in the *worst case*. As shown by Massar *et al.* [175], however, in order to bound the amount of communication, an *infinite* amount of shared randomness is required. More results and lower bounds of communication were achieved by others (see, *e.g.*, References [65, 91, 142, 181, 215, 237]).

Qualitative nature of hidden influences

We look at two qualitative aspects: resistance to noise and fundamental implications of finite speed hidden influences.

Hidden influences are fine tuned

Wood and Spekkens [252] analysed quantum Bell non-local correlations using causal-discovery algorithms from the field of machine learning. These algorithms take as input probability distributions (correlations) and infer the causal structures that lead to these correlations. Causal structures are modelled as follows. Random variables are represented by nodes on a directed acyclic graph. If one variable is the cause of another, then the nodes corresponding to these two variables are connected by an directed edge that leaves the cause. The model parameters are (conditional) probability distributions, each of which represents the probabilities of an effect given all its causes, *i.e.*, a node Q without any cause is modeled by a probability distribution P_Q, and a node R with the causes $S, T, U, \ldots$ is modeled by the probability distribution $P_{R|S,T,U,\ldots}$. Wood and Spekkens show that all inferred causal graphs that reconstruct quantum Bell non-local correlations are *fine tuned* (see Figure 2.5). A causal graph is called *fine tuned* if

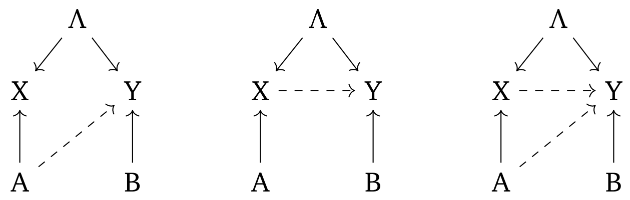


Figure 2.5. Directed acyclic graphs (DAGs) leading to Bell non-local correlations. The variables A and B are the measurement settings for Alice and Bob, respectively. The variables X and Y denote their measurement outcomes. The hidden variable is denoted by Λ. Directed edges represent the causal connections; each edge points to the effect. The hidden influences are represented by dashed edges. The model parameters for the left-most DAG are P_A, P_B, P_Λ, $P_{X|A,\Lambda}$, and $P_{Y|A,B,\Lambda}$. All three DAGs are *fine-tuned*, *i.e.*, disturbing the conditional probability distributions associated to the edges makes the correlations signaling.

under a slight change of the conditional probability distribution, *e.g.*, if they are subjected to noise, the original correlations are lost. In particular, by introducing noise, the *correlations become signaling* at a speed faster than light — and would contradict Einstein's theory of relativity.

Hidden influences cannot have a finite speed

Now, we discuss setups where the hidden influences have a *finite* velocity $v < \infty$ that is greater than the speed of light. Such a model is called *v-causal model*.

We follow the argument of Coretti, Hänggi, and Wolf [84]. Consider a three-party setup with the parties Alice, Bob, and Charlie, where all parties are space-like separated (note that this is the smallest number of parties where such an argument is possible [22]). Furthermore, assume that Bob resides somewhere between Alice and Charlie. If Alice and Bob were to experience Bell non-local correlations, the hidden influence explanation tells us that a hidden signal is sent, without loss of generality, from Alice to Bob. Now, if Bob and Charlie experience Bell non-local correlations as well, then, again without loss of generality, Charlie's system influences Bob's in a hidden way. This gives a configuration in which both, Alice and Charlie, act before (in a preferred reference frame) Bob (see Figure 2.6). However, there exist two-party distributions for Alice-Bob and Bob-Charlie, such that every three-party *non-signaling* distribution between Alice, Bob, and Charlie consistent with the two-party marginals must *necessarily* be Bell non-local for the pair Alice-Charlie. This, however, is inconsistent with the v-causal model: The influence from Alice to Charlie (or *vice versa*) would have to travel at a speed larger than v. By repeating this argument, every *finite* velocity for hidden influences can be ruled out. If one, nevertheless, holds on to a v-causal model, then the assumption that the three-party distribution is *non-signaling* must be dropped: v-causal

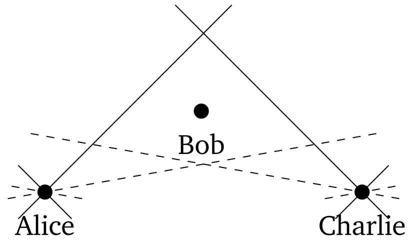


Figure 2.6. The solid lines represent the light cones of Alice and Charlie. The dashed lines represent the v-influence cones, with $c < v < \infty$, of Alice and Charlie.

models are *signaling*.

Unfortunately, the correlations found that lead to this argument are *beyond* those achievable by quantum systems. Similar conclusions in more elaborated settings were established for quantum correlations (see References [22, 24, 221]).

2.1.5 Saving local realism: Relative, retro, and emergent causality

This section deals with other models that can be claimed *local realistic*, and that — in stark contrast to the local hidden-variable models — *can describe Bell non-local correlations*. These models are put in two categories: *relative causality* and *retro causality*. After discussing these, we suggest another way out: *emergent causality*.

Relative causality

The works of Grete Hermann, a philosopher-scientist, have mostly been overlooked. Besides that she spotted a mistake in von Neumann's [243] "proof" that quantum theory cannot be extended to yield deterministic predictions, she gave philosophical arguments against such an extension, and contributed towards a better understanding of *causality*. The latter led her to formulate the measurement process in similar way to the well-known *relative-state interpretation* [111, 112] of quantum theory. This, and a series of other work in the same spirit, are what we elaborate on here. They can all be summarized under the term *relative causality* for reasons that will become clear later.

In a less-known article, Hermann [152] discusses the *apparent* "acausality" of quantum theory. At the beginning of the 20th century, many were disturbed by quantum theory since it does not predict actual measurement outcomes, but probability distributions of these outcomes only. This has led some to conclude that the causal chain within quantum-mechanical processes is broken, or that the theory is incomplete. Hermann addresses these conclusions and argues *against both* of them. We start with the latter.

Following Bohr [51], Hermann's [157] intuition for why only probabilistic predictions are possible, is based on the wave-particle dualism. The trajectory of a particle can be fully described within the "particle-picture," and the propagation of waves within the “wave-picture.” Quantum-mechanical systems can be understood as “particles” or “waves,” depending on the experimenter’s setup and interest of measurement. However, the dynamics *cannot* be described in both pictures *simultaneously*: The pictures are exclusive. So, one pictures contains information which is not contained in the other and *vice versa*. By that, every picture in itself cannot fully account for the outcomes (this has been called quantum complementarity [51] — *dialectic physics* [87]).

Quantum theory merges both pictures (wave, particle) into one. In the standard interpretation, then again, we jump from the “quantum picture” to the “classical picture,” whenever a measurement is made. Within the quantum picture, all dynamics is predicted deterministically by the Schrödinger equation (*before* any measurement). The classical picture, then again, is deterministic and describes the evolution of the apparatus *after* the measurement. The *location* of the jump, which is also known as the Heisenberg cut, is ambiguous: It could be at the first interaction with the measurement apparatus, within the measurement apparatus, when the experimenter reads out the result, *etc.* It is precisely this ambiguity which, according to Hermann [152], forbids one to extend quantum theory:

> “Die Unmöglichkeit, diesen Schnitt eindeutig festzulegen, schließt nun jede Erwartung aus, durch die Entdeckung vorläufig noch unbekannter Qualitäten die Unbestimmtheit der Quantenmechanik zu überwinden.”[7]

Suppose that we *fix* the location of the Heisenberg cut of some measurement process, and that we find some additional variables not present in our current theories at this cut. Since we can shift the cut, these variables would either have to enter the quantum or the classical picture. Both pictures *per se*, however, already deterministically describe the evolution;

> “[j]ede Erweiterung würde zu einer Überbestimmung führen und dadurch mit den bisherigen Ergebnissen der physikalischen Forschung in Konflikt geraten.”[8]

Hermann does not only rule out such additional variables, but additionally shows that in every instance, *all causes* for the upcoming state can be identified — even when a measurement happens. Since *all causes* can be identified, the theory does not provide any room for additional variables to represent “hidden” causes. So, by Hermann, quantum theory *is complete*. In her 1948 article [152], she describes the process of measuring an electron’s position that is described quantum mechanically. That electron interacts with a photon (described quantum mechanically as well), and the result of the interaction, is

[7]“The impossibility to uniquely determine the cut now eliminates any expectation to overcome the uncertainty of quantum mechanics by the discovery of currently unknown qualities.”

[8]“any extension would lead to an overdetermination and by that conflict the hitherto existing results in physics.”

> "eine neue Wellenfunktion, die eindeutig durch die beiden gegebenen Wellenfunktionen [...] bestimmt ist. Diese Wellenfunktion ist eindeutig bestimmt: sie enthält also nicht die Unbestimmtheit, die wir jenem mystischen [...] [P]rozess zuschreiben müßten."[9]

At this stage, one would have to *measure* this new wavefunction to determine the position of the electron.

> "Ohne solche neuen Beobachtungen führt der quantenmechanische Formalismus zu einer immer weitergehenden, aber ganz unanschaulichen Verflechtung der Elementarteilchen."[10]

According to Hermann, this means that

> "[e]rst relativ zu der neuen Messung wird der Zustand des Elektrons nach seinem Zusammenstoß mit dem Lichtquant durch eine Wellenfunktion mit scharfer Ortsangabe [...] beschrieben[,]"[11]

and in that perspective (the key of the argument)

> "bildet es also unmittelbar nach dem Zusammenstoß mit dem Lichtquant durchaus ein für sich bestehendes, durch seine eigene Wellenfunktion charakterisiertes physikalisches System."[12]

In other words, after the interaction, the measured system is in a wavefunction of its own *relative* to the measurement outcome. And that measurement outcome, is what she considers as *cause* for the electron to be at that position. Such a cause, however, cannot be brought in to possibly give better predictions, as it is only accessible to the experimenter *after* the measurement. In another work [155], she writes:

> "Zu einer Voraussage [...] wären jene Gründe trotzdem nicht zu gebrauchen; denn auch sie bestimmen [...] das System nur relativ, und zwar relativ zu der Beobachtung, die bei der Messung selber erst gemacht wurde. Sie konnten also dem Physiker erst nach dieser Beobachtung zur Verfügung stehen und ihm somit keine Vorausberechnung von deren Ergebnis gestatten."[13]

[9] "a new wavefunction which is uniquely determined by the specified wavefunctions. This resulting wavefunction is uniquely determined: it therefore does not contain the uncertainty that we would have to attribute to that mystic process."

[10] "Without any such new observations, the quantum-mechanical formalism leads to a progressing unimaginative braiding of the fundamental particles."

[11] "the electron, after interacting with the photon, is described by a wavefunction with a sharp position only relative to the new measurement,"

[12] "it therefore constitutes an autonomous physical system characterized by its own wavefunction immediately after the interaction with the photon."

[13] "These causes could not have been used for predictions; they determine the system in a relative way, relative to the observation which was obtained only at the moment of the measurement. These causes, therefore, could be accessed after this observation only, and therefore do not allow to predict the outcome."

By these arguments, Hermann does not only describe a predecessor of Everett's relative-state formalism, but also *frees* the notion of causality from the notion of *predictability*.

We briefly conclude Hermann's view. All causes, even if a measurement is performed, can be identified. In the case of a measurement, the state of the measured system is deterministically determined *relative* to the measurement apparatus. Thus, the cause to find the electron at a given position is given *relative* to the state of the apparatus (the cause is the state of the measurement apparatus). Such a cause cannot be used to make better predictions; it is only accessible *after* the measurement itself. This, however, does not mean that quantum processes are "acausal." The notion of "cause" should not be confused with the notion of "predictability."[14]

What Hermann describes (see also Reference [156]) is basically Everett's [111, 112] relative-state formalism — without making use of any formulas. A measurement entangles the object that is measured to the apparatus, and thereafter *relative statements only* are allowed. This, can now be cast into "local realism:" All actions are local and all quantities real — *relatively*. Basically, the wavefunction of some measured system is autonomous *relative to* the wavefunction of the measurement apparatus. Everett goes beyond Hermann's view by explicitly asking for a wavefunction for the *whole universe* (a single quantum universe). Everett's interpretation has often been interpreted as "parallel worlds:" Whenever a system entangles with the apparatus, all alternative results manifest themselves in *parallel* universes — a view advocated by DeWitt [95] and Deutsch [94]. This, however, is a "left-over of classical conceptions. The coexisting branches [...] can only be related to 'worlds' described by classical physics" [168]. To continue in the words of Lévy-Leblond [168]: "[T]he [...] meaning of Everett's ideas is not the coexistence of many worlds, but on the contrary, the existence of a single quantum one." If we fall back to classical concepts, then *relative* statements only, as commented on above, can be made.

We briefly comment on two extensions of the Hermann-Everett interpretation. One is *parallel lives* by Brassard and Raymond-Robichaud [54]: Instead of the universes splitting, the experimenters split — inside a bubble —, and those experimenters are allowed to talk to each other where the quantum predictions are retrieved. This model, just as the Hermann-Everett interpretation, is *local* as well as *realistic*.

If one additionally incorporates "time" into the description, then a *timeless* wavefunction of the whole universe can be constructed. This endeavor has been undertaken by Page and Wootters [195] (see also Reference [254]). There, the state of some parts of the universe is determined *relative* to the reading of another system that is termed "clock." By that, all dynamics (the Schrödinger equation) can be cast in *static form*: *Relative* to the clock, the systems undergo the quantum dynamics.

[14] In some sense, the process that Hermann describes is some kind of self-referential perception [92].

Retro causality

Another approach for a *local-realistic* description of Bell non-local correlations is via *retro causation*, *i.e.*, causation from the future to the past. One such model is called "Parisian zigzag" [88, 89] (see also Reference [206]). Assume a Bell non-local experiment as described above, where Alice and Bob, each, get a photon to be measured according to some measurement setting. In that model, the photons "do not possess polarizations of their own," but rather "borrow one later" [89]. That is, when Alice performs the measurement on her photon, the photon gains a random polarization, that is sent to the past, to the source where the photons were generated. From there on, the "borrowed" polarization travels into the future to Bob (that is why this model is called *zigzag*). Thus, in that model, "Einstein's prohibition to 'telegraph into the past' does not hold at the level" [89] of the photons, but at the level of macroscopic objects only.

A crucial point of this model is that no photon travels *directly* from one party to the other (a path that is "physically empty"). Instead, it travels "along the Feynman-style zigzag [...] made of two timelike vectors (which is physically occupied)."

Emergent causality

Besides *retro causality* and *relative causality*, we speculate on another approach to design a local-realistic model for Bell non-local correlations. In the Leibnizian [76] view, space and time are relational, *i.e.*, quantities are related to each other resulting in these notions. That means, in strong contrast to Newton's view, if no objects would exist, then space and time would exist neither. So what, if time, its direction, and causality are derived from such relations? In the chapter on causality, we formulate such an idea and use it for the studies in the subsequent chapters.

2.2 Relativity theory: Closed time-like curves

Einstein, Lanczos, Gödel, and others showed that general relativity admits space-time geometries with *closed time-like curves* (CTCs). CTCs are world-lines that are space-time periodic; an object traveling on such a world-line bumps into its *younger* self.

> "[I]f P, Q are any two points on a world line of matter, and P precedes Q on this line, there exists a time-like line connecting P and Q on which Q precedes P; *i.e.*, it is theoretically possible in these worlds to travel into the past, or otherwise influence the past" [131].

By influencing the past, we then again can influence the future, *etc.*, or can we not? In the affirmative case, a CTC describes a causal loop. No CTCs have yet been observed in nature and their existence is highly debatable for several reasons. Nevertheless, their existence, for instance in a remote, to us unaccessible, region in the universe, is not ruled out. This motivates the study of causal loops with the aim to better understand their dynamics — if they would exist.

2.2.1 History

We make an attempt to give a rather lengthy account of the history of CTCs — for two reasons. The first being that the discovery of CTCs in general relativity is often overlooked and attributed to Gödel only, and secondly, because these early thoughts on CTCs come with some peculiarities.

To our knowledge, Lanczos was the first to present a solution to Einstein's equations of relativity that exhibits CTCs. This finding, however, as said above, is mostly attributed to Gödel (*e.g.*, see Rindler [214]). In 1924, Lanczos [165] describes a solution to Einstein's equations of relativity where

> "die Zeit keine Koordinate [...], die von $-\infty$ bis $+\infty$ variiert, [zu bedeuten braucht]."[15]

Instead, in his solution to the equations,

> "kann [Zeit] auch eine periodische Koordinate, eine Art Winkelkoordinate darstellen. Es besteht also die Möglichkeit, daß die Welt nicht nur in räumlicher, sondern auch in zeitlicher Beziehung periodisch, also nach allen Richtungen geschlossen ist."[16]

Lanczos' universe has the shape of a cylinder, and is rotationally symmetric; just as the Gödel universe, which was (re)discovered 25 years later. His results inclined him to the following speculation.

> "Da das Elektron nirgends weder Anfang noch Ende haben kann, müßte es nach Ablauf einer Zeitperiode wieder in denselben Raumpunkt zurückgekehrt sein, von wo es ausgegangen, was höchst unwahrscheinlich wäre. Es könnte aber auch seine Existenz in einem anderen Raumpunkt fortsetzen, wobei es scheinbar ein zweites Elektron bilden würde. So kommen wir auf den Gedanken, das vielleicht die Weltschläuche sämtlicher Elektronen nur Ausschnitte aus einem einzigen Urschlauch sind und die einzelnen Elektronen nur zeitlich verschiedene Entwicklungsstadien eines einzigen Urgebildes. Die prinzipielle Gleichheit aller Elementarbausteine der Materie (wenigstens was die negativen Elektronen anbelangt) könnte

[15] "time does not have to be represented by a coordinate that varies from $-\infty$ to $+\infty$."

[16] "time could also be described by a periodic, a kind of an angular coordinate. It is therefore possible that the world is periodic not only with respect to space, but also with respect to time; and so it would be closed in all directions."

> dadurch seine natürliche Erklärung finden und unsere Überzeugung von der Einheit des Universums um große Ausblicke bereichert werden."[17]

Lanczos' opinion on CTCs is that CTCs "aber nirgends zu einem inneren Widerspruch führen."[18]

Einstein mentioned CTCs 10 years *before* Lanczos; without having found a solution to his equations of relativity that would contain them. At that time, Einstein was still working on the generalization of the theory of special relativity — the final stage of the general theory was not yet reached. Einstein [99] wrote:

> "In der gewöhnlichen Relativitätstheorie ist jede Linie, welche die Bewegung eines materiellen Punktes beschreiben kann, [...] notwendig eine ungeschlossene [...]. Das Entsprechende kann in der hier entwickelten Theorie nicht behauptet werden. Es ist daher *a priori* eine Punktbewegung denkbar, bei welcher die vierdimensionale Bahnkurve des Punktes eine fast geschlossene wäre. In diesem Falle könnte *ein und derselbe* materielle Punkt in einem beliebig kleinen raum-zeitlichen Gebiet *in mehreren voneinander scheinbar unabhängingen Exemplaren* vorhanden sein. Dies widerstrebt meinem physikalischen Gefühl aufs lebhafteste. Ich bin aber nicht imstande, den Nachweis zu führen, daß das Auftreten solcher Bahnkurven nach der entwickelten Theorie ausgeschlossen sei."[19]

Later, Einstein asked Carathéodory in two letters to look for a solution of the equations of general relativity that contain CTCs. In 1916, Einstein [104] wrote:

> "Wollen Sie nicht noch etwas über das Problem der geschlossenen Zeitlinien nachdenken? Hier liegt der Kern des noch ungelösten Teiles des Raum-Zeit-Problems."[20]

[17]"Since for the electron there is no beginning and end, it would have to return to the same location, from where it started, after a period of time, which, however, would be highly unlikely. Then again, its existence could continue in another point in space, and by that apparently form another electron. By that we arrive at the idea that the paths of many electrons are just sections from a path of a single electron, and that the apparent electrons are just different stages of development of an *ur*-electron. The principle indistinguishability of electrons can therefore be explained in a natural way. This would enrich the credo of the unity of the universe."

[18]"do nowhere lead to an inner contradiction."

[19]"In ordinary relativity theory, every line that might represent the motion of a material point is necessarily unclosed. The analogous statement for the here developed theory cannot be claimed. Therefore, a movement of a point is *a priori* imaginable, where the four-dimensional curve would almost be closed. In that case, in an arbitrary small space-time region, *one and the same* material point could coexist *in multiple apparently independent copies*. This contradicts my feeling for physics in the most vivid sense. However, I am not capable to show that the occurrence of such curves is ruled out in the final theory."

[20]As translated in the English translation supplement: "Wouldn't you like to reflect a bit more on the problem of closed time lines? Here lies the core of the as yet unresolved portion of the space-time problem."

Einstein's demand remained unanswered. Later in the same year, Einstein [105] insisted:

> "Wenn Sie aber die Frage nach den geschlossenen Zeitlinien lösen, werde ich mich mit gefalteten Händen vor Sie hinstellen.... Hier steckt etwas dahinter, des Schweisses der besten würdig."[21]

Also here, Carathéodory did not get into this at all.

Furthermore, Weyl [246] commented on CTCs in his book *Raum Zeit Materie*:

> "Von jedem Weltpunkt geht der Doppelkegel der aktiven Zukunft und der passiven Vergangenheit aus. Während in der speziellen Relativitätstheorie diese durch ein Zwischengebiet getrennt sind, ist es hier an sich sehr wohl möglich, daß der Kegel der aktiven Zukunft über den der passiven Vergangenheit hinübergreift; es kann also prinzipiell geschehen, daß ich jetzt Ereignisse miterlebe, die zum Teil erst eine Wirkung meiner künftigen Entschlüsse und Handlungen sind. Auch ist es nicht ausgeschlossen, daß eine Weltline, obschon sie in jedem Punkte zeitartige Richtung besitzt, insbesondere die Weltline meines Leibes, in die Nähe eines Weltpunktes zurückkehrt, den sie schon einmal passierte. Daraus würde dann ein radikaleres Doppelgängertum resultieren, als es je ein E. T. A. Hoffmann ausgedacht hat. [...] So Paradoxes da zutage kommt, ein eigentlicher Widerspruch zu den in unserem Erleben unmittelbar gegebenen Tatsachen tritt nirgendwo hervor."[22]

At this point we would like to point at three peculiarities on which some of Einstein, Weyl, and Lanczos seem to agree. (1) All three of them mention that such curves could lead to "Doppelgänger" — multiple copies of the same object — that reside next to each other in space. (2) From (1) the step to *closed* lines seems not large: An object could just take over the place of its "Doppelgänger." Nevertheless, Einstein and Weyl did not directly address *closed* lines. (3) Weyl and Lanczos commented that such world lines are *not contradictory*. Einstein did not explicitly mention the issue of contradiction.

[21] As translated in the English translation supplement: "If you solve the problem of the closed time lines, though, I shall place myself before you with hands folded in reverence.... Behind this is something worthy of the swat of the best of us."

[22] As translated by Brose [247]: "Every world-point is the origin of the double-cone of the active future and the passive past. Whereas in the special theory of relativity these two portions are separated by an intervening region, it is certainly possible in the present case for the cone of the active future to overlap with that of the passive past; so that, in principle, it is possible to experience events now that will in part be an effect of my future resolves and actions. Moreover, it is not impossible for a world-line (in particular, that of my body), although it has a time-like direction at every point, to return to the neighbourhood of a point which it has already once passed through. The result would be a spectral image of the world more fearful than anything the weird fantasy of E. T. A. Hoffmann has ever conjured up. [...] Although paradoxes of this kind appear, nowhere do we find any real contradiction to the facts directly presented to us in experience."

Moreover, Einstein did not employ this issue of contradictions to rule out such world lines from his theory. From (2), however, the thought of the grandfather antinomy (see Section 1.2) could immediately be reached.

2.2.2 Gödel on closed time-like curves

Gödel's [135] first article on CTCs appeared 25 years *after* Lanczos'. His initial description of CTCs, which appeared in 1949, is a purely philosophical one, where he understands Einstein's theory as defending epistemological idealism. The mathematical version (see Gödel [131]) appeared in the same year as a birthday gift to Einstein — in a special issue of the journal Reviews of Modern Physics dedicated to Einstein's birthday. In the mathematical version, he gives a solution to the equations of relativity that allow for CTCs. The universe he describes is very similar to Lanczos' and van Stockum's [242]. In the same year, Gödel [134] held a lecture on rotating universes, which was published posthumously. One follow-up appeared in 1952 [133].

Einstein's [102] reply to Gödel is rather short and addresses the question on the *reversibility* of the dynamics in his theory; there is no *a priori* notion of *future* and *past*. But if we consider a process that is *signaling*, then one event necessarily precedes the other:

> "Wesentlich ist hierbei, daß das Senden eines Signals ein *nicht umkehrbarer* Prozess ist im Sinne der Thermodynamik, ein Prozess, der mit dem Wachsen der Entropie verknüpft ist (während *nach unserem gegenwärtigen Wissen* alle *Elementarprozesse* reversibel sind)."[23]

Contrary to Einstein, Weyl, and Lanczos (see peculiarities above), Gödel added a note in the German translation of the philosophical article [132] to clarify some points. He wrote:

> "Auch der Entropiesatz dürfte mit den obigen Lösungen durchaus verträglich sein. Denn man kann in ihnen für alle zeitartigen Linien in eindeutiger und stetiger Weise eine positive Richtung definieren. Ferner ist es unendlich unwahrscheinlich, daß irgendein materielles System *genau* in sich zurückläuft. Wenn das aber nur angenähert geschieht, so bedeutet es bloß, daß irgendwo 2 Exemplare desselben Systems (im allgemeinen mit verschiedenen Entropiewerten) gleichzeitig nebeneinander vorhanden sind."[24]

[23] This was translated by Paul Arthur Schlipp [106]: "What is essential in this is the fact that the sending of a signal is, in the sense of thermodynamics, an irreversible process, a process which is connected with the growth of entropy (whereas, *according to our present knowledge*, all elementary processes are reversible)."

[24] "Also the second law of thermodynamics might thoroughly be in agreement with the above solution. The reason for this is that one can define an unambiguous and continuous positive direction for every time-like line. Furthermore, it is infinitely improbable that some material system *exactly* goes back onto itself. If, however, this happens only approximatively, then it means that somewhere 2 copies of the same system (generally with different entropies) simultaneously exist next to each other."

Even though in his article of the same year he [131] describes *closed* time-like curves, his *addendum* above says it is *infinitely improbable* for a system to travel on such a *closed* word-line. The last sentence in Gödel's *addendum* reads:

> "Die Anfangsbedingungen sind allerdings in solchen Welten nicht vollkommen frei wählbar."[25]

Thus, the issue of *contradictory* dynamics (the grandfather antinomy) is resolved by — what is known as Novikov's self-consistency principle (see Section 5.2) — *forbidding* a time-traveler to kill his or her grandfather. That is, if some closed time-like curves exist in the future of some space-like surface, then the initial conditions set at that surface are *restricted* to initial conditions that do *not lead to a contradiction.* This point has later been addressed and — in some sense — resolved by Morris *et al.* [182].

There are also voices *against* the physical existence of CTCs. For instance, Reichenbach [211] suggests that "we must make the assumption that *there are no closed causal chains*." In the same spirit, Hawking [149] conjectures that "[t]he laws of physics do not allow the appearance of closed timelike curves." The motivation for these conjectures is that CTCs seem eligible to the grandfather antinomy (see Section 1.2).

2.3 Quantum gravity: Hardy's approach

Our two currently most successful theories are quantum theory and general relativity. Whilst these theories are used widely, the search for a unified theory is ongoing. This unified theory would be a theory of quantum gravity. Many attempts to merge both theories have been conducted (see, *e.g.*, Reference [216]). Hardy's [145–148, 174] approach to a unified theory is the following. Quantum theory is a *probabilistic* theory (at least, *apparently* probabilistic — see Section 2.1) where a *fixed* notion of space-time is assumed. The latter means, that quantum theory — in its current form — is compatible with some background time. General relativity, then again, is a *deterministic* theory where space-time is *dynamic*. Hardy now suggests that a candidate for quantum gravity should be *probabilistic* with a *dynamic* space-time; such a formulation of quantum gravity would uphold the *relaxed* features from quantum theory and general relativity.

Towards a formulation of quantum gravity, Hardy introduced the *causaloid* framework — a mathematical framework for that very purpose, in which many theories can be cast so they become probabilistic and admit an *indefinite* causal structures. In such a theory one could imagine events in space-time, and the space-time coordinates themselves to be entangled, leading to quantum effects on that footing. This necessitates a new treatment of *causality* — a motivation for this thesis.

The *process-matrix* framework, a framework for *quantum* correlations that allows *indefinite* causal structures, was later developed by Oreshkov, Costa, and Brukner [194].

[25] "The initial conditions in such worlds, however, are not completely freely selectable."

Another framework where *indefinite* causal structures might be allowed is the *quantum combs* framework [68–71, 81]. Our work closely follows the process-matrix framework.

Chapter 3

On causality

> The law of causality, I believe, like much that passes muster among philosophers, is a relic of a bygone age, surviving, like the monarchy, only because it is erroneously supposed to do no harm.
>
> Bertrand Russell [219]

The notion of causality has been (and remains to be) highly discussed. The English word *causality* has its roots in the Latin word *causa*, which means *reason, cause*. Thus, the notion of causality aims at pinpointing to some reason. Closely related to the word *causality* is the conjunction *because* which means *be*ing the *cause* of. We do not give a full account of the notion of *causality* — which is rather impossible (we redirect the interested reader to *The Oxford Handbook of Causation* [39]). Instead, we focus on the *cause* and *effect* relation. A discussion of different approaches to causality can be found in the beforehand mentioned textbook. In physics, causality was understood (up to the beginning of the 20th century) as being provided by the physical laws in terms of *deterministic* laws (see, *.e.g.*, Reference [196]).[1] The laws were (and sometimes still are) used to deterministically predict the configuration of particles, objects, *etc.* at some time t. Since quantum theory was formulated, however, this notion gets questioned frequently. A first *doubt* on "causality" came when people realized that quantum theory *does not* fully predict measurement results — only probabilities can be derived from the formalism. As Grete Hermann [154] put it:

> "Das physikalische Ergebnis der Quantenmechanik, von dem die Erschütterung altgewohnter naturphilosophischer Auffasungen, insbesondere der

[1] With the development of *relativity* theory, the notion of "causality" is sometimes used to refer to the light-cones structure.

> Kausalvorestellungen ausgeht, besagt, daß der Vorausberechnung künftiger Naturvorgänge eine scharfe, unüberwindbare Schranke gezogen ist."[2]

This issue was resolved by *freeing the notion of causality from perfect predictability* [157] (see also Section 2.1.5 of this thesis). The quantum community was shocked again by the works of Einstein, Podolsky, and Rosen [107] and Bell [40]: Not only that the *actual* outcome of a measurement cannot be predicted by quantum theory, but any extension of the formalism (under some assumptions) does not allow one to predict the results (see also References [82, 83]).

3.1 Definition of causal relations

> In Mente nulla est absoluta, sive libera voluntas; sed Mens ad hoc, vel illud volendum determinatur a causa, quae etiam ab alia determinata est, & haec iterum ab alia, & sic in infinitum.[3]
>
> Baruch de Spinoza [90]

This section aims at giving some definitions around the notion of causality. These definitions are used in the following chapters. We model physical quantities by random variables. In this thesis, we understand causality as *causal relations* between random variables.

Closely related are *space-time relations*. Here, we use the theory of special relativity as a representative theory to discuss them. Suppose we are given a number of random variables where every random variables is located within space-time. The future of a random variable A is the space-time region that light could reach from A. This region is also called the future light cone of A. The past of A, then again, is the space-time region from which light could have arrived.

Definition 3 (Space-time past, space-time future)**.** A random variable A (B) is in the *space-time past (future)* of the random variable B (A), if and only if a signal at the speed of light or slower could travel from A to B. This relation is expressed by $A \preceq_{\mathrm{ST}} B$. If neither $A \preceq_{\mathrm{ST}} B$ nor $B \preceq_{\mathrm{ST}} A$ holds, then we write $A \not\preceq_{\mathrm{ST}} \not\succeq_{\mathrm{ST}} B$.

Note that special relativity does *not* provide a distinction between past and future light cones; the theory is symmetric in the time direction. Therefore, the above definition relies on a *postulated direction of time*.

[2] As translated in Reference [158]: "The physical consequence of quantum mechanics, which casts doubt upon traditional views in the philosophy of nature, especially upon the concept of causality, implies that the predictive calculation of future processes in nature is restricted by a sharp, insurmountable limit."

[3] In the mind there is no absolute or free will; but the mind is determined to wish this or that by a cause, which has also been determined by another cause, and this last by another cause, and so on to infinity.

The following notion is aimed to reflect the relation between a *cause* and an *effect* as opposed to the notion of *space-time past* and *space-time future*.

Definition 4 (Causal past, causal future, case, and effect)**.** A random variable A (B) is said to be in the *causal past (future)* of the random variable B (A), if and only if A and B are correlated and A is *free*. This relation is expressed by $A \preceq B$. The random variable A is called a *cause* and B is called an *effect*.

In this definition, we distinguish between *free* and *non-free* random variables. At this point we do not aim to give an explanation of what a free variable is, but just *postulate* the existence of both types of random variables. One interpretation of *free* variables is that their value can be set directly by an experimenter, *e.g.*, a light switch as opposed to the light bulb that is controlled by the switch.[4]

If the relation $A \preceq B$ holds, then we can send a signal from A to B. Based on relativity theory (without closed time-like curves), the relation $A \preceq B$ implies $A \preceq_{\mathrm{ST}} B$. The converse, however, does not need to hold: The event of deers mating on the other side of lake Lugano one second after we finished a chess game is within the future light cone of our action, but unrelated. We can express that setting with random variables: Place two *independent* random variables, A and B, in space-time such that $A \preceq_{\mathrm{ST}} B$.

We adopt the above definition of causal past and causal future to *sets* of random variables.

Definition 5 (Causal past and causal future for sets)**.** The set $\mathcal{S}$ ($\mathcal{T}$) is said to be in the *causal past (future)* of the set $\mathcal{T}$ ($\mathcal{S}$), if and only if there exist random variables $A \in \mathcal{S}$, $B \in \mathcal{T}$ such that $A \preceq B$ and where the relation $A' \npreceq B'$ holds for every $A' \in \mathcal{S}$, $B' \in \mathcal{T}$. This relation is expressed by $\mathcal{S} \preceq \mathcal{T}$.

3.1.1 Freeness from space-time relations and correlations

By Bell [43], freeness of variables means "that the values of such variables have implications only in their future light cones." This standpoint was taken by Colbeck and Renner [78] to give a definition of *free variables*. A random variable A is called *free* if it is *uncorrelated* to all random variables *outside* of its future light cone (see Figure 3.1). However, their approach to distinguish between free and non-free variables can only be applied in a setting where "[t]he space time structure has been taken as given" [42]. This is also reflected by the fact that Colbeck and Renner [78] do not refer to usual random variables, but call them *space-time variables*.

3.1.2 Causal relations from freeness and correlations

Instead of defining *freeness based on the space-time relations* of the variables, we can define the *causal relations based on freeness*. Suppose you are given two random variables, A and X, where you know that A is free and X is not free (*e.g.*, a knob and a

[4] This follows the *interventionists* or *agent-based* approaches to causality [253].

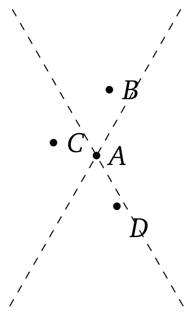


Figure 3.1. The upper cone is the future light cone of the random variable A. The random variable A is free if it is uncorrelated to C and D.

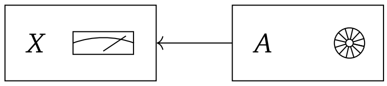


Figure 3.2. If the random variable A is correlated to X and if A is free, then A is in the causal past of X.

gauge). Now, if A is correlated to X, then by manipulating A we can manipulate X. So, a random variable A is called to be in the *causal past* of another random variable X if A is free and correlated to X (see Definition 5 and Figure 3.2). Thus, by knowing which variables are free and which ones are not, and by knowing the correlations among the variables, we can deduce the causal relations between them. Since this work considers *relaxations* of causal relations, we mainly stick to this second approach, and use the first approach for comparison. By doing so, we place the notion of freeness as *more fundamental* when compared to the notion of causal relations (see Table 3.1).

This analysis of causality suggests a third approach, which is too speculative, and hence not discussed beyond the following sentence. One could, as opposed to the two approaches above, try to introduce some *subjective* notion of *freeness* that depends on the state of the observer, and derive in that way *subjective* causal relations from the correlations.

	Postulated	**Derived**	**from**
Colbeck and Renner [80]	Causal structure	Freeness	Correlations
Here	Freeness	Causal structure	Correlations

Table 3.1. Two approaches: One to derive *freeness* from correlations and a postulated *causal structure*, and the other to derive the *causal structure* from correlations and postulated *freeness*.

3.2 Causal correlations and causal inequalities

The models we study are of *operational* nature: Parties interact with systems and the resulting correlations are studied. Before we define causal correlations (correlations from space-time settings) and causal inequalities (inequalities satisfied by such correlations), we introduce the notion of *parties*.

Definition 6 (Party and causal relation between parties)**.** A *party* $S_j = (A_j, X_j, L_j)$ is a triple that consists of a *free* random variable A_j, a *non-free* random variable X_j, and a local operation L_j. The variable A_j is called *input of* S_j, and the variable X_j is called *output of* S_j. For two parties S_j and S_k, we say that S_j (S_k) is in the causal past (future) of S_k (S_j) if and only if $A_j \preceq X_k$. This relation is expressed by $S_j \preceq S_k$.

Note that a setting $S_j \preceq S_k \preceq S_j$ is not ruled out by that definition. The local operation L_j of a party S_j is not discussed further here, but is discussed in the next chapter. In later parts of this work, we also make use of parties with more than two random variables. A (free) random variable of a party can also be composed of multiple (free) random variables, *e.g.*, $A_j = (A'_j, A''_j, A'''_j)$.

Causal correlations are correlations among random variables of parties that can be simulated in a setting where every party is located in a space-time point, and where any action of a party can have an influence in her future light cone only. For two parties, this can be formalized easily.

Definition 7 (Two-party causal correlations)**.** Let S_1 and S_2 be two parties. The probability distribution $P_{X_1,X_2|A_1,A_2}$ is called *causal* if and only if it can be written as

$$P_{X_1,X_2|A_1,A_2} = pP_{X_1|A_1}P_{X_2|X_1,A_1,A_2} + (1-p)P_{X_1|X_2,A_1,A_2}P_{X_2|A_2}\,,$$

where p is a probability.

This definition (illustrated in Figure 3.3) reads that with probability p, party S_1 is in the *past light cone* of party S_2 ($S_1 \preceq_{\mathrm{ST}} S_2$). In that case, party S_2 cannot send any signal to S_1 (the variables X_1 and A_2 are uncorrelated), and conversely, S_1 can signal to S_2 (X_2 can depend on A_1). Note that sometimes such a decomposition is not unique, *e.g.*, in the case where all random variables are independent.

For more than two parties, the definition becomes more subtle. The reason for this is that if we assume that a party S_j can influence *all events* happening in his or her future light cone, then S_j can also have an influence on the *locations* of the parties in that light cone (see Figure 3.4). We give a recursive definition, which can be found in a recent article by Abbott *et al.* [8]. This subtlety was also discussed in other articles [28, 193].

Definition 8 (Multi-party causal correlations)**.** Let $S_1, S_2, \dots, S_n$ be parties. The probability distribution $P_{X_1,X_2,\dots,X_n|A_1,A_2,\dots,A_n}$ is called *causal* if and only if it can be written

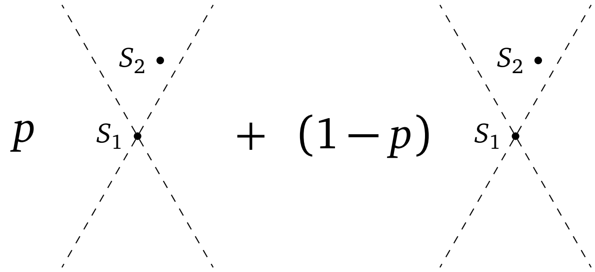


Figure 3.3. The relation $S_1 \preceq_{\mathrm{ST}} S_2$ holds with probability p, and otherwise, the relation $S_2 \preceq_{\mathrm{ST}} S_1$ holds.

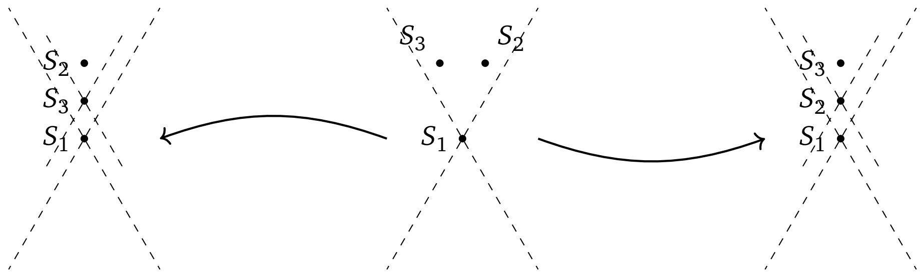


Figure 3.4. Party S_1 can influence her future light cone, and by that position the parties *within her future* to reflect either structure.

as

$$P_{X_1,X_2,\ldots,X_n|A_1,A_2,\ldots,A_n} = \sum_{j=1}^{n} p_j P_{X_j|A_j} Q_j \,,$$

where p_j are probabilities, and where, for every j, Q_j is a *causal distribution*

$$P_{X_1,X_2,\ldots,X_{j-1},X_{j+1},\ldots,X_n|A_1,A_2,\ldots,A_{j-1},A_{j+1},\ldots,A_n} \,.$$

By this definition, party S_j is in the *past light cone* of every other party with probability p_j. In that event, party S_j can send signals to and set the locations of *all* other parties in his or her future light cone. Without loss of generality, one can assume that party S_j positions the remaining $n-1$ parties in such a way that there is another party (S_k) in the past of the rest $n-2$. This party S_k, then again, can signal to and set the locations of the parties in his or her future light cone, *etc.*

A *causal inequality* is an inequality of probability expressions that is satisfied by *causal* distributions only. This means that if a distribution leads to a *violation* of a causal inequality, then that distribution is *not* causal, *i.e.*, it cannot be decomposed as in Definition 8. A causal inequality depends on the setting, *i.e.*, the *number* of parties and the *dimensions* of the input and output per party. Free variables are assumed to be uniformly distributed.

For a fixed setting, the set of all causal correlations forms a polytope [193]. Thus, every face of such a polytope is a causal inequality. For two parties where every party has a binary input and a binary output, all inequalities have been enumerated [53]. The same has been done recently for the simplest setting with three parties [8]. In this "simplest" setting, whenever a party has input 0, then he or she outputs a constant 0, otherwise he or she can output either 0 or 1.

We briefly take a look at an example of a causal inequality. This inequality is constructed for two parties $S_1 = \{A_1 = A, X_1 = X\}$ and $S_2 = \{A_2 = (B, B'), X_2 = Y\}$, where all inputs and outputs are binary. Assume, that the inputs are uniformly distributed. Every causal distribution in that setting satisfies

$$p^{\text{OCB}} := \frac{1}{2}\left(\Pr\left(X = B \mid B' = 0\right) + \Pr\left(Y = A \mid B' = 1\right)\right) \leq \frac{3}{4}. \tag{3.1}$$

This example is taken from the seminal article by Oreshkov, Costa, and Brukner [194]. The inequality reads that if the random variable B' takes the value 0, then party S_1 has to guess the free variable B of party S_2. For this guess to be correct with a probability that is *strictly* larger than 1/2, party S_2 has to be in the past of S_1 ($S_2 \preceq_{\text{ST}} S_1$). In the case where the variable B' takes the value 1, party S_2 has to guess the free variable of party S_1, and thus the relation $S_1 \preceq_{\text{ST}} S_2$ should hold. For causal distributions, the highest value for this expression is 3/4. The reason for this is that one party can, in the best case, send the value of his or her free variable to the other party. But then, this other party can at best just make a random guess. Any ordering ($S_1 \preceq_{\text{ST}} S_2$, or $S_1 \succeq_{\text{ST}} S_2$, or $S_1 \npreceq_{\text{ST}} \nsucceq_{\text{ST}} S_2$), and thus also any probabilistic mixture of these orderings, does not lead to a higher success probability (value of p^{OCB}).

3.3 Causal loops

The notions of *cause* and *effect* (see Definition 5) can be understood as *device independent* notions. By that we mean that the definition is applied on the level of random variables, and not on the *mechanism* bringing about the correlations, or how these random variables "interact."

If one does not presuppose some *background* direction of *causation* — or some *definite* order — then it is imaginable that a theory could incorporate some *causal loops*. In a causal loop, some values are derived (depending on the theory) from others, which then again are used to calculate the former. This is not to be understood as in a *feedback loop*, where the values of some variables change *over time*. First of all, here we do not refer to time *at all*, and secondly, the variables take just *one* value.

The notion of a *causal loop* is in that sense symmetric, and thus, it cannot be inferred directly from Definition 5. To give a definition for what a *causal loop* is, we hence depart from the *device independent* view and consider some *underlying* mechanism.

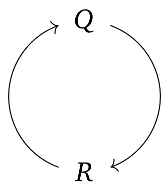


Figure 3.5. A causal loop, where some stochastic process transforms Q to R, and back to Q.

Definition 9 (Causal loop)**.** Let Q and R be two non-free random variables. If in a theory, some stochastic process computes R from Q, and some (possibly other stochastic process) computes Q from R, then we call the composition of both processes a *causal loop* (see Figure 3.5). In the *deterministic* case, the random variables are replaced by *variables* and the stochastic processes by *functions*.

In the above definition the (random) variables do not "successively" take *different* values; rather, they are fixed by some process or function. So, the variable R from Figure 3.5 is the output of some process that depends on Q, and Q is the output of some other process that depends on R — *simultaneously*. Such loops, however, come with some logical antinomies.

3.3.1 Antinomies

We discuss two *logical problems — antinomies —* of causal loops. Both antinomies are illustrated by time travel. By going to the past, and interacting in the past, one creates a causal loop: The past state depends on the current, and the current then again on the past one.

One antinomy is called *grandfather antinomy*. Suppose someone travels to the past, where his or her grandfather is alive yet where the grandfather has not yet met his beloved one (the grandmother). Once this person has found the grandfather, he or she *kills the grandfather*. By doing so, the grandfather would not have been involved in a series of acts that would have led to the birth of this time-traveling person. So, this person would have never been born. But then, he or she could also not travel to the past to kill the grandfather. But if the grandfather was not killed, then this person is alive, but ... — a *logical contradiction*.

In terms of the definition of causal loops above, the grandfather antinomy arises when *no consistent* assignment of values to the variables exists. As an example (see Figure 3.6a), consider two binary variables R and Q, where R is the output of the *identity* function applied to q, and q is the output of the *bit-flip* function applied to r.

The other antinomy is called *information antinomy*. This antinomy is slightly more subtle, but as we are going to see, its consequences would undermine all scientific thoughts. Suppose someone wakes up one morning to find a book on his or her desk.

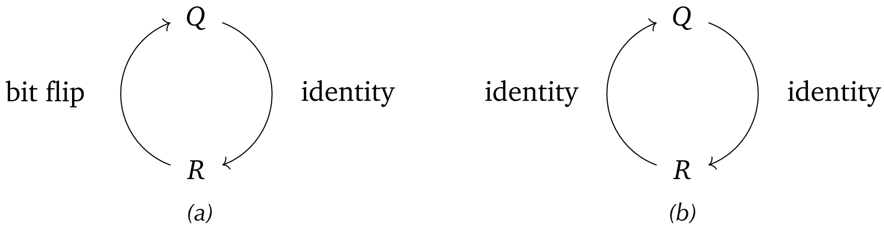


Figure 3.6. (a) Schematic description of the grandfather antinomy. (b) Schematic description of the information antinomy.

After going through the stages of happiness and boredom, he or she publishes the book, wins some prices, invents a time machine, travels back in time and places that very same copy of the book on his or her desk, just the night before he or she has found it. Now, the question is, who or what wrote the book, and more concretely, why is the content of the book not different? The strings of text and numbers seem to arise *ex nihilo*, where no reason exists for its structure to be like it is. As Deutsch [93] puts it:

> "It is a fundamental principle of the philosophy of science that the solutions of problems do not spring fully formed into the [u]niverse, *i.e.*, as initial data, but emerge only through evolutionary or rational processes. In adopting this evolutionary principle we reject such antirational doctrines as creationism."

Thus, the solution to problems, or complex information (randomness taken aside), could not simply "pop up," but must be the result of some process. Otherwise, as Deutsch remarks it, science would follow the doctrine of *creationism*.

By reconsidering the definition of causal loops above, the information antinomy arises when the processes do *not specify* the values of the variables. Then, we do not know what values they should take. As an example, consider both operations that connects R with Q, and Q with R to be the *identity* functions (see Figure 3.6b).

Chapter 4

Correlations without causal order

> [. . .] on s'imagine des *Places*, des *Traces*, des *Espaces*, quoyque ces choses ne consistent que dans la verité des *Rapports*, & nullement dans quelque realité absolue.[1]
>
> Gottfried Wilhelm Leibniz [75]

Logical consistency does *not* imply causal correlations. This, and related results, are shown in the current chapter.

In Chapter 2, we elaborated on an emergent notion of causal order, and on causal loops. Here, we pose the question of whether local assumptions and logical consistency only imply causal correlations. Causal correlations are defined in Chapter 3.

The study of correlations without causal order was initiated by Oreshkov, Costa, and Brukner [194]. They showed that non-causal correlations (see Definition 8) are achievable in a model where quantum theory holds locally, but no assumption on the causal order of the parties is made. In the same work (see also Reference [85]), it was shown that in the classical limit, *i.e.*, when all operations are restricted to operations in a *fixed* basis, the resulting correlations for two parties are necessarily *causal*. This leads to the impression that a *causal order* might emerge from a quantum-to-classical transition. An extension of that result was shown by Baumann and Brukner [27] (see also Reference [26]): If both parties measure their respective quantum systems in a *fixed* basis, then the resulting correlations can always be simulated causally (their output systems do not have to be classical). However, for three or more parties, this *appearance* of a causal order from the quantum-to-classical transition *vanishes*. This is shown in the current chapter. After we state the assumptions, we derive the mathematical framework: the framework for correlations without causal order. We continue by briefly interpreting the framework, and then by characterizing the processes appearing in the

[1] As translated by Clarke [74] himself: "[. . .] Men fancy *Places*, *Traces*, and *Spaces*; though those things consiste only in the Truth of *Relations*, and not at all in any absolute Reality."

framework in two ways: by polytopes and by fixed-point theorems. Having done that, we explicitly state some *non-causal* correlations, and show that they remain achievable even if one presumes that all dynamics must be *reversible*. Furthermore, such reversible dynamics leading to non-causal correlations can be cast in the quantum framework (the process-matrix framework), where the process matrix represents a *unitary* transformation — *reversible* quantum dynamics. Finally, we show that the developed framework is *equivalent* to the quantum framework in the *classical limit*.

4.1 Assumptions

The assumptions for the model of classical correlations without causal order are listed here. Usually, theories, *e.g.*, quantum theory, are formulated with the assumption of a *global causal order*. This assumption fixes the relative positions of the parties. We depart from this view by *dropping* that often made (sometimes implicit) assumption. This means that for some parties $S_1, S_2, \dots$ we do not assume some causal order $S_1 \preceq S_2$ or $S_1 \succeq S_2$ *etc.* among the parties. Instead, we base the framework on the following list of *local* assumptions only. What is *outside* of the parties will be called *environment*. The environment will then allow the parties to interact in the most general way. At this point, the environment is held purely mathematical; possible interpretations of the environment are given in the subsequent Section 4.3.

(C) The underlying systems the parties interact with are *classical*, *i.e.*, random variables. Thus, in the most general setting, the parties can interact with the random variables in an *arbitrary* way, as long as the interaction maps random variables to random variables. The latter implies that any such interaction is governed by probability theory, and that a party has an input as well as an output. So, every party has a direction of time *within* her laboratory.

(Isol) Every party is *isolated* from all other parties. This means that no *joint* operations on random variables can be performed; every party can perform operations on her own random variables only. The reasons for why parties must be isolated are that local interactions only are possible, and that otherwise the model would be trivial: *All* correlations become possible.

(1-Int) Parties interact with the environment only *once*. The interaction is held as general as possible. This means, a party obtains a random variable from the environment, manipulates it, and outputs a random variable to the environment.

(LC) The assumption of *logical consistency* states that the distribution over all non-free random variables conditioned on all free random variables is a map f which is *multi-linear* in the choice of the local operations. This means that, for *any* choice of local operations, the map f gives a probability distribution (the probabilities

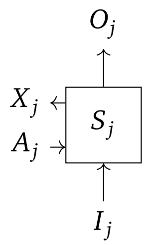


Figure 4.1. A party S_j has a *free* variable A_j and a channel L_j that transforms A_j and I_j (the input from the environment) to X_j and O_j (the output to the environment).

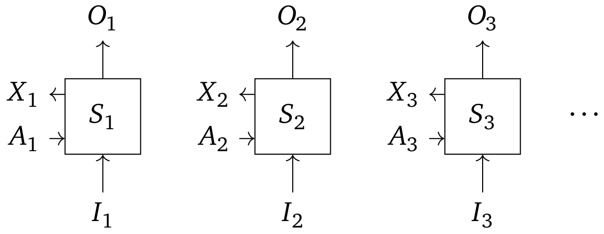


Figure 4.2. Parties are isolated and therefore, interact independently with the environment.

sum up to unity and are non-negative) over all non-free random variables conditioned on all free random variables. The requirement for the map to be *multilinear* is of operational nature. Assume one could choose an operation out of two to perform, and that for every operation, one receives a different output. If a coin-flip chooses the operation, the output should then be distributed according to the distribution of that coin.

4.2 Mathematical framework

The underlying states with which the parties interact are random variables (see Assumption (C)). In the most general setting respecting Assumption (C) and Assumption (1-Int), a party S_j *first* receives a system I_j from the environment and *thereafter* outputs a system O_j to the environment. Thus, we can make the following definition.

Definition 10 (Local operation). The local operation L_j of a party S_j is $P_{X_j,O_j|A_j,I_j}$. The set of all local operations of S_j is $\mathcal{L}_j$.

A local operation is a stochastic map, *i.e.*, a conditional probability distribution that has the form $P_{X_j,O_j|A_j,I_j}$ (see Figure 4.1). Due to the assumption that the parties are isolated (see Assumption (Isol)), *multiple* parties are represented by a *set* of parties $\{S_1, S_2, \dots\}$ (see Figure 4.2). We drop the indices to refer to a collection of objects, *e.g.*, X stands for $(X_1, X_2, \dots)$. Assumption (LC) asks for a *multi-linear* map f from the local operations $L_1, L_2, \dots$ to the probability distribution $P_{X,O,I|A}$.

Definition 11 (Map)**.** The map

$$\begin{aligned} f : \mathcal{L} &\to \mathcal{G}\,, \\ (L_1, L_2, \dots) &\mapsto P_{X,O,I|A}\,, \end{aligned}$$

where f is linear in $L \in \mathcal{L}$, generates the probability distribution of all non-free variables conditioned on the free variables.

Theorem 1 (Probability expression)**.** *The map f is*

$$f : (L_1, L_2, \dots) \mapsto P_{I|O} L_1 L_2 \dots\,.$$

Before we state the proof, we consider a simpler scenario: We ignore the free variables (they can be embedded into the local operations) and we marginalize over the non-free random variables $X_1, X_2, \dots$. Thus, we consider the restricted local operations $L'_j = P_{O_j|I_j}$ only, and ask for the most general multi-linear map f':

$$\begin{aligned} f' : \mathcal{L}'_1 \times \mathcal{L}'_2 \times \dots &\to \mathcal{G}'\,, \\ (L'_1, L'_2, \dots) &\mapsto P_{I,O}\,, \end{aligned}$$

where $\mathcal{L}'_j$ is the set of all local operations for party S_j of the form $P_{O_j|I_j}$, and where $\mathcal{G}'$ is the set of the probability distributions $P_{I,O}$.

Proof of Theorem 1. If a party S_j applies the local operation $L_j(o,i) = \delta_{o,0}$, *i.e.*, if S_j deterministically outputs 0, then the probability $P_{I,O}(i,k)$ for $k \neq 0$ should be 0. Otherwise, the joint probability $P_{I,O}$ would not reflect the probabilities observed by the parties. Therefore, we can rewrite the function f' as

$$\begin{aligned} f' : (\mathbb{R}, \mathbb{R}, \dots) &\to \mathbb{R}\,, \\ (P_{O_1|I_1}(o_1, i_1), P_{O_2|I_2}(o_2, i_2), \dots) &\mapsto P_{I,O}(i,o)\,. \end{aligned}$$

That f' is multi-linear means that $f'(L'_1(o_1,i_1), L'_2(o_2,i_2), \dots)$ equals

$$\begin{aligned} c(o,i) &+ e_1(o,i) L'_1(o_1,i_1) + e_2(o,I) L'_2(o_2,i_2) + \dots \\ &+ e_{1,2}(o,i) L'_1(o_1,i_1) L'_2(o_2,i_2) + \dots \\ &+ e_{1,2,\dots}(o,i) L'_1(o_1,i_1) L'_2(o_2,i_2) \dots\,, \end{aligned}$$

where $c(o,i)$ is some constant and where $e_{p,q,r,\dots}(o,i)$ is the coefficient for the product of probabilities of the parties $S_p, S_q, S_r, \dots$. As above, we fix the local operations of all parties to be

$$L'_j(o_j, i_j) = \begin{cases} 1 & o_j = 0\,, \\ 0 & \text{otherwise,} \end{cases}$$

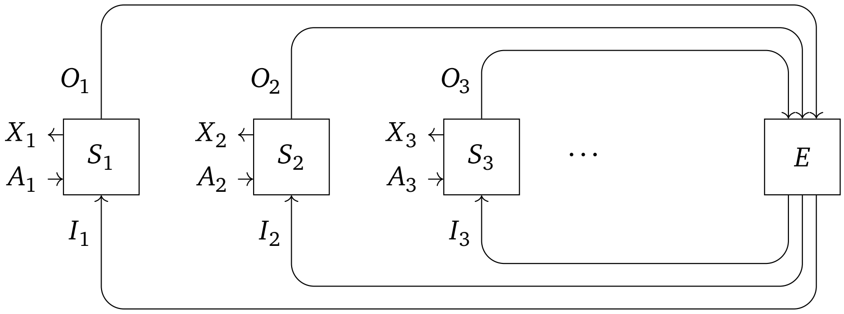


Figure 4.3. The environment E is a channel that takes the outputs of the parties and produces the inputs to the parties.

i.e., every party deterministically sets the random variable O_j to the value 0. As discussed at the beginning of the proof, $P_{I,O}(i,o)$ for $o \neq (0,0,\dots)$ should be 0, which implies that $c(o,i)$ must be 0. Then again, if at least one party uses the local operation from above, then $P_{I,O}(i,o)$ for $o \neq (0,0,\dots)$ should remain 0. This implies that all coefficients of strict subsets of parties must be zero. Thus, we are left with

$$P_{I,O}(i,o) = e_{1,2,\dots}(o,i)L'_1(o_1,i_1)L'_2(o_2,i_2)\dots) .$$

If we fix the parties to deterministically output o', then the condition that probabilities sum up to unity implies

$$\sum_{i,o} e_{1,2,\dots}(o,i)\delta_{o_1,o'_1}\delta_{o_2,o'_2}\dots = \sum_i e_{1,2,\dots}(o',i) = 1 .$$

By repeating this argument for all o', we get

$$\sum_i e_{1,2,\dots}(o,i) = 1 .$$

The coefficient $e_{1,2,\dots}(o,i)$ can thus be interpreted as a conditional probability distribution $P_{I|O}(i,o)$. □

The conditional probability distribution $P_{I|O}$ can be interpreted as channel E (see Figure 4.3). We call this channel *environment* or *classical process*.

Definition 12 (Environment, classical process)**.** The *environment*, also called *classical process* E is a channel $P_{I|O}$ that takes the random variables $O_1, O_2, \dots$ as input from the respective parties, and produces the random variables $I_1, I_2, \dots$ as outputs, which serve as inputs *to* the respective parties.

4.2.1 Examples of classical processes

We briefly discuss three examples with three parties S_1, S_2, S_3, where all random variables are assumed to be binary.

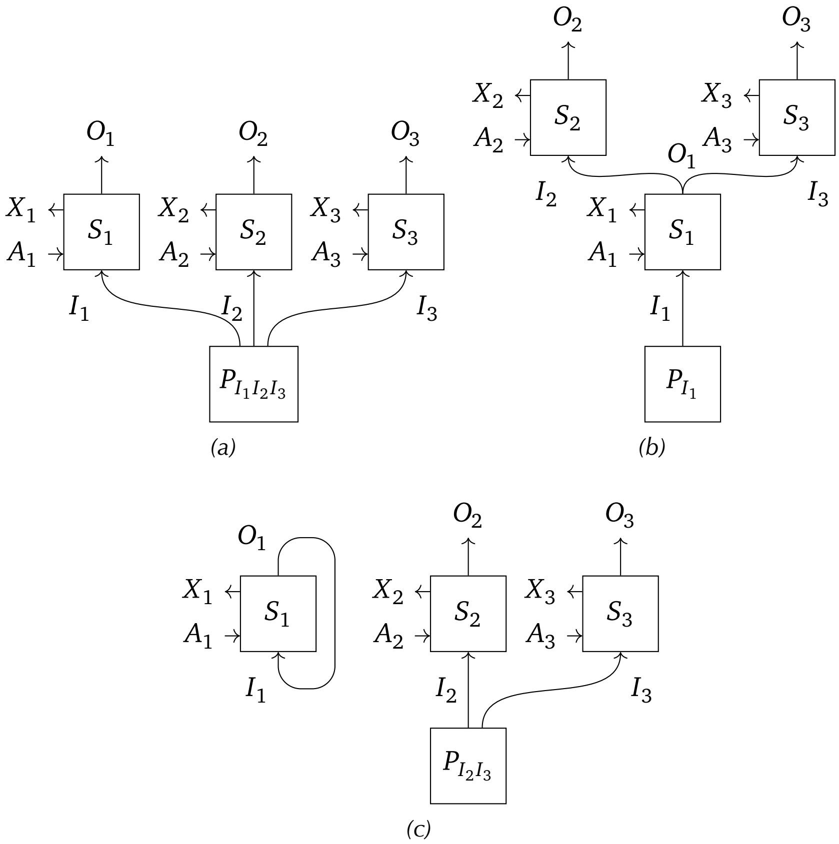


Figure 4.4. (a) The environment E as a *state*. (b) The environment E as a channel from S_1 to S_2, S_3. (c) The environment E produces a *loop* from the output of S_1 to the input of the same party.

Example 1. An example of an environment is a *state* P_{I_1,I_2,I_3}, *i.e.*, a channel that ignores the inputs O_1, O_2, O_3:

$$P_{I_1,I_2,I_3|O_1,O_2,O_3} = P_{I_1,I_2,I_3} .$$

This can be visualized as in Figure 4.4a.

Example 2. Another example is an environment that gives a state to party S_1 and that sends the output of S_1 to all other parties:

$$P_{I_1,I_2,I_3|O_1,O_2,O_3} = P_{I_1} P_{I_2,I_3|O_1} ,$$

with

$$P_{I_2,I_3|O_1}(i_2, i_3, o_1) = \delta_{i_2,o_1} \delta_{i_3,o_1} .$$

Such an environment generates the causal relations $S_1 \preceq S_2$ and $S_1 \preceq S_3$. This is shown in Figure 4.4b.

Example 3. A final example (see Figure 4.4c) is an environment that deterministically forwards the output of party S_1 to the input of S_1, and where all other parties receive a state:

$$P_{I_1,I_2,I_3|O_1,O_2,O_3} = P_{I_1|O_1} P_{I_2,I_3} ,$$

with

$$P_{I_1|O_1}(i_1, o_1) = \delta_{i_1,o_1} .$$

Note that this example has a *logical* problem. If party S_1 chooses to perform the local operation

$$L_1(x_1, o_1, a_1, i_1) = P_{X_1,O_1|A_1,I_1}(x_1, o_1, a_1, i_1) = \delta_{x_1,0} \delta_{o_1,i_1 \oplus 1} ,$$

i.e., she flips the input I_1 to produce the output O_1, then no *consistent* assignment of values to the variables is possible: If S_1 receives the value a from the environment, flips it, *i.e.*, she outputs the value $a \oplus 1$, which then again is forwarded to her input, then we get the inconsistency that a has to be equal $a \oplus 1$, which is not possible. This is an instantiation of the grandfather antinomy (see Section 3.3.1).

4.2.2 Logical consistency

Because of logical problems as in Example 3, we need to restrict the environment E in such a way that *for any choice of local operations* no logical problem arises. This follows from Assumption (LC) combined with Assumption (C), which states that the parties can perform *arbitrary* operations.

Definition 13 (Logical consistency)**.** An environment $E = P_{I|O}$ is called *logically consistent* if and only if for any choice of local operations of the parties, the expression

$$P_{I|O}P_{X_1,O_1|A_1,I_1}P_{X_2,O_2|A_2,I_2}\cdots$$

yields a *probability distribution* $P_{X,O,I|A}$.

Since we anyhow consider *all* choices of local operations, we can, for the sake of simplicity (as also done above), ignore the random variables $A_1,A_2,\dots$ and the random variables $X_1,X_2,\dots$. Thus, the definition of an environment E to be logical consistency simplifies to:

$$\forall \left(P_{O_1|I_1},P_{O_2|I_2},\dots\right) \in \mathcal{L}_1' \times \mathcal{L}_2' \times \cdots : P_{I,O} = P_{I|O}P_{O_1|I_1}P_{O_2|I_2}\cdots ,$$

where the operations $P_{O_j|I_j}$ are the *reduced* local operations of the parties — the inputs A_j and outputs X_j are ignored. This is the same as saying that for any choice of *reduced* local operations of the parties, the following two conditions are satisfied:

$$\sum_{o\in\mathcal{O},i\in\mathcal{I}} P_{I,O}(i,o) = 1 ,$$

$$\forall o \in \mathcal{O}, i \in \mathcal{I} : P_{I,O}(i,o) \geq 0 .$$

The first condition requires that the probabilities *sum to unity*, and the second conditions asks for *non-negative* probabilities.

The environment of Example 3, as opposed to the other two examples, is logically *in*consistent. To see this inconsistency take the local operation $L_1'(o_1,i_1)$ for party S_1 to be $\delta_{o_1,i_1\oplus 1}$. By this, we get

$$\sum_{o\in\mathcal{O},i\in\mathcal{I}} \delta_{i_1,o_1}P_{I_2,I_3}(i_2,i_3)\delta_{o_1,i_1\oplus 1}P_{O_2|I_2}(o_2,i_2)P_{O_3|I_3}(o_3,i_3) = 0 ,$$

which, however, should be 1.

4.3 Interpretation of the environment

There are several ways to interpret the environment E. Throughout this work we discuss some of them in detail.

One way to interpret the environment is that the causal relations among the parties are not defined *a priori*. Rather, they are defined via the correlations that the environment and the choice of local operations establish. This advocates the view that physics is not "played" on a space-time stage; the stage *emerges* from the correlations. This could be a called *Leibnizian* interpretation; space-time is relational.

This contrasts the view that the parties have *fixed* locations within space-time. In that case, the environment can be thought of as a *back-in-time* channel in the sense of a closed time-like curve. We call this interpretation *Einsteinian*. We elaborate more on this in Chapter 5.

Finally, we could interpret the environment *logically*. Then, it is not an *actual process* that happens. Rather, the environment serves as a *tool* that helps to describe the most general correlations if one follows the assumptions from Sections 4.1. We follow this logical interpretation in Chapter 6, where we discuss it from a point of view of computer science.

4.4 Characterization with polytopes

Polytopes and their properties are described in the preliminaries (see Appendix). The objects used in Section 4.2 can be represented by probability vectors and stochastic matrices (see Appendix as well).

The environment E can be modeled by a stochastic matrix $\hat{E} = \hat{P}_{I|O}$ of dimension $|\mathcal{I}| \times |\mathcal{O}|$. The local operation L_j of a party S_j, then again, can be modeled by a stochastic matrix $\hat{P}_{X_j,O_j|A_j,I_j}$. As above, we restrict ourselves in the following to the *reduced* local operations: $\hat{P}_{O_j|I_j} =: \hat{L}'_j$. The dimensions of the matrix $\hat{L}'_j$ depends on the cardinality of the sets $\mathcal{O}_j$ and $\mathcal{I}_j$: $\hat{L}'_j$ is a $|\mathcal{O}_j|$ by $|\mathcal{I}_j|$ matrix. In the following, the set $\mathcal{L}'_j$ refers to the set of all stochastic matrices of that form. As above, the set $\mathcal{L}'$ (where we drop the index j) is the set of the Kronecker products of all stochastic matrices for all parties involved, *i.e.*, $\mathcal{L}' = \mathcal{L}'_1 \times \mathcal{L}'_2 \times \ldots$. Thus, an element $\hat{L}' \in \mathcal{L}'$ has the form $\hat{L}'_1 \otimes \hat{L}'_2 \otimes \ldots$ where $\hat{L}'_j \in \mathcal{L}'_j$. We adopt the definition of logical consistency (see Definition 13) to the current representation, where the probabilities are expressed by

$$P_{I|O}(i,o) = \left(\vec{i}_1 \otimes \vec{i}_2 \otimes \cdots\right)^T \hat{E}\,(\vec{o}_1 \otimes \vec{o}_2 \otimes \cdots)\,, \tag{4.1}$$

$$P_{O_j|I_j}(o_j,i_j) = \vec{o}_j^{\,T} \hat{L}'_j \vec{i}_j\,. \tag{4.2}$$

The condition for an environment E to be logically consistent is stated in the following theorem.

Theorem 2 (Logical consistency). *An environment E is logically consistent if and only if*

$$\forall \hat{L}' \in \mathcal{D} : \mathrm{Tr}\left(\hat{E}\hat{L}'\right) = 1\,, \tag{4.3}$$

$$\forall (m,n) \in \mathcal{I} \times \mathcal{O} : \hat{E}_{m,n} \geq 0\,. \tag{4.4}$$

The set $\mathcal{D} \subset \mathcal{L}'$ is the set of all deterministic *local operations from the set $\mathcal{L}'$.*

The first condition states that for any choice of *deterministic* operations of the parties, the probabilities sum up to unity. The second condition states that the probabilities are non-negative.

Proof of Theorem 2. By plugging in Equations (4.1) and (4.2), the total-probability condition is

$$\forall \hat{L}' \in \mathcal{L}' : \sum_{i,o} \left(\vec{i}^T \hat{E} \vec{o}\right) \cdot \left(\vec{o}_1^T \hat{L}_1' \vec{i}_1\right) \cdot \left(\vec{o}_2^T \hat{L}_2' \vec{i}_2\right) = 1\,.$$

This can now be rewritten as

$$\forall \hat{L}' \in \mathcal{L}' : \sum_{i,o} \vec{i}^T \hat{E} \vec{o} \vec{o}^T \hat{L}' \vec{i} = 1\,,$$

from which we obtain

$$\forall \hat{L}' \in \mathcal{L}' : \mathrm{Tr}\left(\hat{E}\hat{L}'\right) = 1\,.$$

Because any local operation can be written as a convex combination of *deterministic* ones, and because the above expression is *linear* in the choice of local operations, it is sufficient to consider the *deterministic* ones only. By this we arrive at the condition (4.3).

The non-negativity condition (4.4) is easy to show. Probabilities must be non-negative. The local operations contain non-negative entries only. Thus, every entry of $\hat{E}$ must be non-negative as well. □

Theorem 2 can also be stated by considering a smaller set of local operations:

Theorem 3 (Necessary and sufficient set for the total-probability condition)**.** *The condition 4.3 can also be stated by restricting every party S_j to the operations*

$$\hat{D}_{j,m,n} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

which is the zero matrix with 1 *entries in the first row, except for the column m: There the entry* 1 *is at position n. Such an operation outputs the all-zero state for any input except for input m, where the output is n. This matrix has dimension $|\mathcal{O}_j| \times |\mathcal{I}_j|$. Formally, this means*

$$\begin{aligned} &\forall (m,n) \in \mathcal{O} \times \mathcal{I} : \mathrm{Tr}\left(\hat{E}\left(\hat{D}_{1,m_1,n_1} \otimes \hat{D}_{2,m_2,n_2} \otimes \dots\right)\right) = 1 \\ &\Longleftrightarrow \\ &\forall \hat{L}' \in \mathcal{D} : \mathrm{Tr}\left(\hat{E}\hat{L}'\right) = 1\,. \end{aligned}$$

Proof. We restrict ourselves to the single-party scenario — the multi-party case follows from linearity. The direction $\Longleftarrow$ is trivial since:

$$\forall (m,n) \in \mathcal{O}_1 \times I_1 : D_{1,m,n} \in \mathcal{D} .$$

For the other direction, we show that any $\hat{L}' \in \mathcal{D}$ can be written as a linear combination of $\hat{D}_{1,m,n}$ with $(m,n) \in \mathcal{O} \times \mathcal{I}$. The proofs then follows from the fact that the probability expression is linear in the choice of operation. So, any $\hat{L}' \in \mathcal{D}$, where k is mapped to a_k can be written as

$$\hat{L}' = \sum_k \vec{w}_{a_k} \vec{v}_k^T ,$$

where $\vec{v}_k$ is the $|\mathcal{I}|$-dimensional vector with a 1 entry at position k and 0s everywhere else, and where the vector $\vec{w}_{a_k}$, then again, is the $|\mathcal{O}|$-dimensional zero vector with a 1 at position a_k. Now, $\hat{L}'$ can be written as

$$\hat{L}' = \sum_{k \in \mathcal{I}} \hat{D}_{1,k,a_k} - (|\mathcal{I}| - 1) \hat{D}_{1,0,0} .$$

□

The family $\{\hat{D}_{j,m,n} | (m,n) \in \mathcal{O}_j \times \mathcal{I}_j\}$ of deterministic operations has size

$$|\mathcal{I}_j| \big(|\mathcal{O}_j| - 1 \big) + 1 .$$

In contrast, the set $\mathcal{D}_j$ has cardinality $|\mathcal{O}_j|^{|\mathcal{I}_j|}$.

4.4.1 Polytope of logically consistent environments

The above characterization for logically consistent environments defines a polytope. Let us assume a setup with n parties $S_1, S_2, \ldots, S_n$. Now, the set of *logically consistent environments E* forms a polytope:

Theorem 4 (Polytope)**.** *The H-representation of the polytope of logically consistent environments without causal order is*

$$\forall (m,n) \in \mathcal{O} \times \mathcal{I} : \mathrm{Tr}\Big(\hat{E}\left(\hat{D}_{1,m_1,n_1} \otimes \hat{D}_{2,m_2,n_2} \otimes \ldots\right)\Big) = 1 ,$$
$$\forall i,j : \hat{E}_{i,j} \geq 0 .$$

If all input and output spaces are isomorphic and have dimension d, then the polytope has d^{2n} facets and dimension

$$d^{2n} - (d(d-1)+1)^n ,$$

which is exponential in the number of parties.

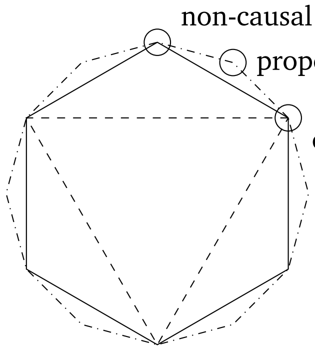


Figure 4.5. Schematic polytopes. The triangle inside represents the set of all *causal* environments. The dash-dotted polytope is the polytope of all environments (non-causal as well), and the polytope in-between is the non-causal polytope restricted to *deterministic* extremal points.

4.4.2 Deterministic-extrema polytope

As is shown later, for some configurations, the polytope of logically consistent environments consists of extremal points that are *probabilistic*. Such extremal points thus are *intrinsically* probabilistic; they cannot be expressed as a convex combination of deterministic extremal points. Rather, they are *consistent* mixtures of *inconsistent* points. We call them *proper mixtures*. This motivates to study the following, restricted polytope:

Definition 14 (Deterministic-extrema polytope). The *deterministic-extrema polytope* is defined as the polytope of logically consistent environments *where all extremal points are deterministic* (see polytope with the solid lines in Figure 4.5).

This polytope is solved by first solving the regular polytope and then by selecting only the deterministic extremal points. Let us define two sets: $\mathcal{D}$ is the set of all environments from the deterministic-extrema polytope, and the set $\mathcal{D}_{\text{ext}}$ is the set of all *extremal points* of that polytope. In the following two chapters, we state a result where these definitions come handy.

4.4.3 Polytopes with binary systems for one to three parties

We discuss four examples where all input and output spaces are binary: the single-party scenario, the two-party scenario, the three-party scenario, and the deterministic-extrema polytope of the three-party scenario. For all examples we use $\mathcal{I}_j = \mathcal{O}_j = \{0,1\}$ for $j \in \{1,2,3\}$. Thus, the set $\mathcal{L}_{\text{det}}$ of deterministic local operations is

$$\mathcal{L}_{\text{det}} = \left\{ \hat{D}_0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \hat{D}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \hat{D}_{\text{id}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \hat{D}_{\text{not}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

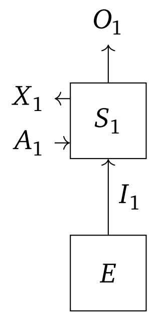


Figure 4.6. In the single-party scenario, the environment is equivalent to a state.

where $\hat{D}_0$ and $\hat{D}_1$ produce the output 0 and 1, respectively, and where $\hat{D}_{\text{id}}$ and $\hat{D}_{\text{not}}$ are the identity and bit-flip operations. Because of Theorem 3, we do not have to consider $\hat{D}_{\text{not}}$.

Example 4 (Single-party scenario)**.** The environment $\hat{E}$ is a two-by-two matrix:

$$\hat{E} = \begin{pmatrix} w_0 & w_1 \\ w_2 & w_3 \end{pmatrix} .$$

With the identities

$$\mathrm{Tr}\left(\hat{E}\hat{D}_0\right) = 1\,, \quad \mathrm{Tr}\left(\hat{E}\hat{D}_1\right) = 1\,, \quad \mathrm{Tr}\left(\hat{E}\hat{D}_{\text{id}}\right) = 1\,,$$

we can eliminate three variables from $\hat{E}$ and get

$$\hat{E} = \begin{pmatrix} w_0 & w_0 \\ 1-w_0 & 1-w_0 \end{pmatrix} .$$

From the non-negativity constraint we get

$$\begin{aligned} w_0 &\geq 0\,, \\ 1-w_0 &\geq 0\,. \end{aligned}$$

This gives a one-dimensional polytope with the extremal points $w_0 = 0$ and $w_0 = 1$. The respective environments are

$$\hat{E}_0 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} , \quad \hat{E}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} .$$

Every convex combination of these two environments is such that it ignores the output of the single party. Thus, in the single-party scenario, an environment is equivalent to a state, as shown in Figure 4.6.

Example 5 (Two-parties scenario)**.** In the two-parties scenario with binary inputs and outputs, the environment as a stochastic matrix $\hat{E}$ is a 4-by-4 matrix where the conditions

$$\forall i,j \in \{0,1,\mathrm{id}\} : \mathrm{Tr}\left(\hat{E}\left(\hat{D}_i \otimes \hat{D}_j\right)\right) = 1\,,$$
$$\forall m,n \in \{1,2,3,4\} : \hat{E}_{m,n} \geq 0$$

are fulfilled. We can eliminate 9 out of the 16 variables. Thus, the polytope is 7-dimensional. There are 12 extremal points:

$$\hat{E}_0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{E}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{E}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{E}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\hat{E}_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{E}_5 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{E}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad \hat{E}_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

$$\hat{E}_8 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{E}_9 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{E}_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad \hat{E}_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

The four first environments represent states (the input state is independent of the output state of the parties). The following four environments represent channels from S_1 to S_2 ($S_1 \preceq S_2$), and the last four environments are channels from S_2 to S_1 ($S_2 \preceq S_1$). These environments are shown in Figure 4.7.

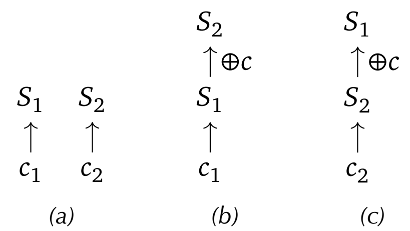


Figure 4.7. The two-parties polytope consists of environments that represent either (a) a state, (b) a (bit-flip) channel from S_1 to S_2, or (c) a (bit-flip) channel in the reverse direction.

Example 6 (Three-parties scenario)**.** Here, the environment $\hat{E}$ is an 8-by-8 matrix. This polytope is given by

$$\begin{aligned} \forall i,j,k \in \{0,1,\text{id}\} : \text{Tr}\left(\hat{E}\left(\hat{D}_i \otimes \hat{D}_j \otimes \hat{D}_k\right)\right) = 1\,, \\ \forall m,n \in \{1,2,\ldots,8\} : \hat{E}_{m,n} \geq 0\,, \end{aligned}$$

and is $64 - 27 = 37$-dimensional. We used cdd+ [124] to enumerate the extremal points. For this setting, the polytope consists of $710'760$ extremal points. Here, as opposed to the two examples looked at before, some extremal points are *fine-tuned*. This means, that such an extremal point *cannot* be expressed as a convex combination of *deterministic* ones from the polytope. Rather, these extremal points are a convex combination of deterministic matrices where some of them are necessarily *inconsistent*. A tiny variation to the mixture of such fine-tuned processes might render them inconsistent. One *fine-tuned* extremal point of this polytope is

$$\hat{E}_{\text{ex1}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{4.5}$$

This environment is a uniform mixture of two *circular* channels. One channel takes the output of S_1 and forwards it to S_2, takes the output of S_2 and forwards it to S_3, and takes the output of S_3 and forwards it to S_1. The other channel is as the one just described with additional bit-flips (see Figure 4.8). The environment $\hat{E}_{\text{ex1}}$, as a

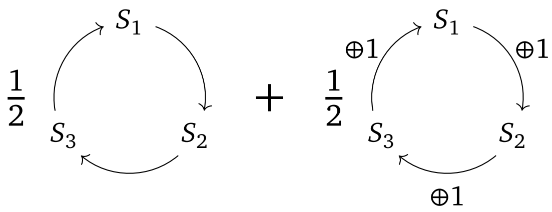


Figure 4.8. The extremal point $\hat{E}_{\text{ex1}}$ corresponds to an environment which is the uniform mixture of two circular channels: the identity and the bit-flip channel.

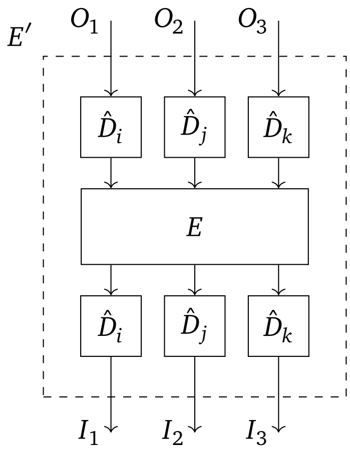


Figure 4.9. From a logically consistent environment E one can always obtain another logically consistent environment E' where $(0,0,0)$ is mapped to $(0,0,0)$. This is done by squeezing E into a set of identity or bit-flip operations.

conditional probability distribution, is

$$P_{I|O}(i,o) = \begin{cases} \frac{1}{2} & \text{for } (i_1,i_2,i_3) = (o_3,o_1,o_2), \\ \frac{1}{2} & \text{for } (i_1,i_2,i_3) = (o_3 \oplus 1, o_1 \oplus 1, o_2 \oplus 1), \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

Example 7 (Three-parties deterministic-extrema scenario)**.** If we restrict ourselves to the deterministic extremal points of the polytope discussed above, we find a total of 744 extremal points. We can exploit some symmetry to reduce the number by a factor of 8: We look only at those extremal points where on input $(0,0,0)$ the environment outputs $(0,0,0)$ again. This is obtained by some relabeling of the bits (see Figure 4.9). Thus, we remain with a total of 93 extremal points. They can be categorized within seven classes (see also Figure 4.10):

1. all parties receive a state (no channels);

2. two-parties to one-party channels;

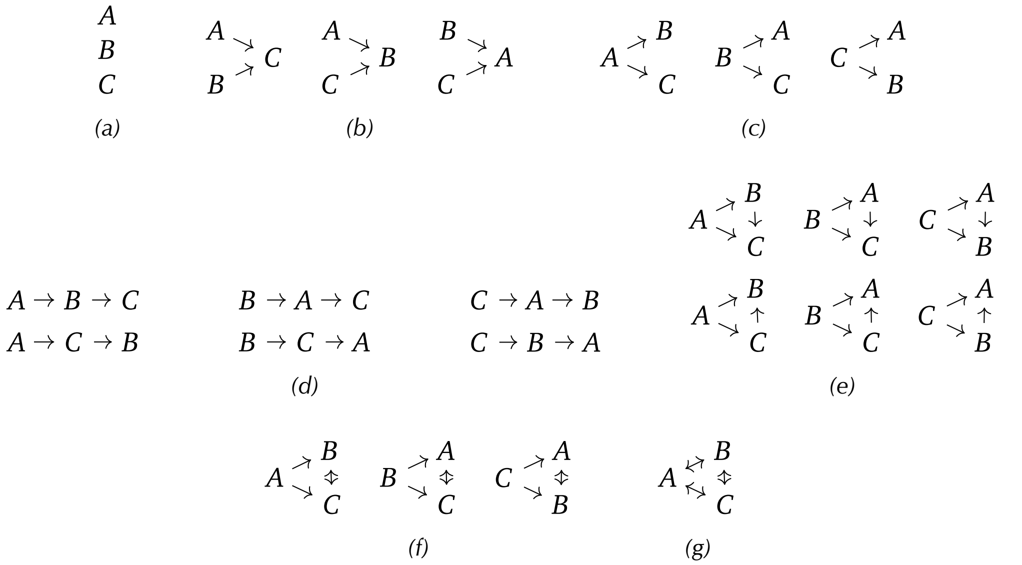


Figure 4.10. All classes of environments in of the deterministic-extrema polytope for three parties and binary inputs and outputs.

3. one-party to two-parties channels;
4. one-party to one-party to one-party channels;
5. one-party to two-parties and one-party to one-party channels;
6. one-party to two-parties and two-way channels between these two parties;
7. every input depends on the other two outputs.

The environments of class 1 are simply states. Up to the symmetry discussed above, only one extremal point belongs to this class. The number of extremal points per class are shown in Table 4.1. The last two classes are of special interest since the respective environments lead to *non-causal* correlations. One extremal point of the last class (which is also an extremal point of the three-parties polytope discussed above) is

$$\hat{E}_{\mathrm{ex2}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.7}$$

Class	Number of extremal points
1	1
2	$3 \cdot 7 = 21$
3	3
4	6
5	$6 \cdot 5 = 30$
6	$3 \cdot 8 = 24$
7	8
Total	93

Table 4.1. Number of deterministic extremal points per class.

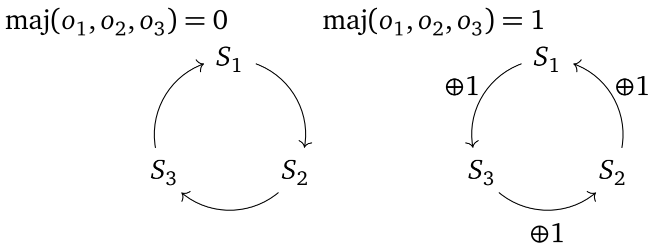


Figure 4.11. The extremal point shown in Equation 4.8 is a *conditional* circular channel.

This is a *deterministic* circular channel which depends on the majority of the inputs. It was initially found by Araújo and Feix [14]. If the majority of the inputs is 0, then the environment implements a clock-wise circular channel. Otherwise, it implements a counter clock-wise circular channel with bit-flips (see Figure 4.11):

$$P_{I|O}(i,o) = \delta_{i_1,\neg o_2 \wedge o_3} \delta_{i_2,\neg o_3 \wedge o_1} \delta_{i_3,\neg o_1 \wedge o_2} . \tag{4.8}$$

In Figure 4.12, we project this and the preceding polytope onto a plane, together with the discussed examples.

4.5 Characterization with fixed points

The property that an environment is logically consistent can be expressed by statements about its fixed points. Basically, an environment is logically consistent if and only if it has a *unique* fixed point. We show this in the deterministic case first, and head on to the probabilistic later.

A conditional probability distribution $P_{R|Q}$ with only $0-1$ entries can be written as a function and *vice versa*.

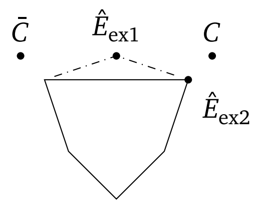


Figure 4.12. The three-parties polytope for binary systems is projected onto a plane. The solid lines show the smaller deterministic-extrema polytope. The points C and $\bar{C}$ represent the circular channels (clock-wise and anti clock-wise).

Definition 15 (Conditional probability distributions and functions)**.** Let $P_{R|Q}$ be a conditional probability distribution. We define the function $f_{P_{R|Q}}$ as

$$\begin{aligned} f_{P_{R|Q}} &: \mathcal{Q} \to \mathcal{R} \\ &q \mapsto r \text{ such that } P_{R|Q}(r,q) = 1\,. \end{aligned}$$

For a function $g : \mathcal{S} \to \mathcal{T}$ we define the corresponding conditional probability distribution $P^g_{T|S}$ as

$$P^g_{T|S}(t,s) = \begin{cases} 1 & \text{if } g(s) = t\,, \\ 0 & \text{otherwise,} \end{cases}$$

where T and S are random variables with sample spaces $\mathcal{T}$ and $\mathcal{S}$ respectively.

By using this definition, we can switch from functions to conditional probability distributions and back.

Theorem 5 (Unique fixed points for deterministic environments)**.** *A deterministic environment E is logically consistent if and only if, for any choice of local operations of the parties, the concatenated function has a* unique *fixed point, i.e.,*

$$\forall L' \in \mathcal{L}', \exists! i \in \mathcal{I} : i = f_E\left(f_{L'_1}(i_1), f_{L'_2}(i_2), \dots\right),$$

where $\exists!$ is the uniqueness quantifier.

Proof. By Theorem 2 we have

$$\forall \hat{L} \in \mathcal{L}_{\text{det}} : \text{Tr}\left(\hat{E}\hat{L}\right) = 1\,.$$

Since $\hat{L}$ and $\hat{D}$ have $0-1$ entries only, so does $\hat{M} := \hat{E}\hat{L}$. That the trace of $\hat{M}$ equals one means that the diagonal $\vec{q}$ of M has a single 1 entry, and that

$$\hat{M}\vec{q} = \vec{q}\,.$$

The position of the 1 entry in $\vec{q}$ corresponds to the fixed point. □

This theorem can be understood in the following way. If there would be *no* fixed point, then we would have a logical inconsistency in the spirit of the grandfather antinomy (see Section 3.3.1). If there would be *more than one* fixed points, then we run into the information antinomy (see Section 3.3.1): What is the actual state observed by the parties? If one would apply some function to reduce the number of fixed points back to one (as is done in Deutsch [93]), then one introduced a *non-linearity* into the dynamics, which leads to other problems.

A similar theorem can be stated for *probabilistic* environments:

Theorem 6 (Average number of fixed points is one)**.** *Let E be a convex combination of deterministic stochastic matrices:*

$$\hat{E} = \sum_k p_k \hat{D}_k \,.$$

Then, for any choice of local operations, we have

$$\forall \hat{L}' \in \mathcal{L}_{det} : \sum_k p_k \left| \left\{ i \mid i = f_{D_k}\left(f_{L'_1}(i_1), f_{L'_2}(i_2), \dots \right) \right\} \right| = 1 \,.$$

Proof. By plugging in some decomposition

$$\hat{E} = \sum_k p_k \hat{D}_k$$

of $\hat{E}$, the total-probability condition becomes

$$\forall \hat{L}' \in \mathcal{L}_{\text{det}} : \text{Tr}\left(\hat{E}\hat{L}'\right) = \text{Tr}\left(\sum_k p_k \hat{D}_k \hat{L}' \right) = \sum_k p_k \, \text{Tr}\left(\hat{D}_k \hat{L}'\right) = 1 \,.$$

We conclude the proof by noting that, as in the proof above, the expression $\text{Tr}\left(\hat{D}_k \hat{L}'\right)$ is the *number of fixed points* of $\hat{D}_k \hat{L}'$. □

This theorem states that the *average* number of fixed points is 1. Note that the transition matrices $\hat{D}_k$ of some decomposition of $\hat{E}$ do not necessarily have to be logically consistent.

4.5.1 Illustrations of the fixed-point theorems

We discuss the two examples (Equations (4.5) and (4.7)), starting with the latter.

Example 8. The environment $\hat{E}_{\text{ex2}}$ (see Equation (4.7) and Figure 4.11) is *deterministic*. The function $f_{\hat{E}_{\text{ex2}}}$ as a function is shown in Table 4.2. If all three parties S_1, S_2, S_3 apply the identity operation ($o_j = i_j$), then the fixed point is $(0,0,0)$. Now, suppose that party S_3 applies the bit-flip operation instead. This is shown in Table 4.3 where we added a third column that contains the outputs of the parties *after* the local operations have been applied. In this case, the fixed point is $(1,0,1)$.

o_1	o_2	o_3	i_1	i_2	i_3
0	0	0	0	0	0
0	0	1	1	0	0
0	1	0	0	0	1
0	1	1	0	0	1
1	0	0	0	1	0
1	0	1	1	0	0
1	1	0	0	1	0
1	1	1	0	0	0

Table 4.2. The function table of $f_{\hat{E}_{\text{ex2}}}$.

o_1	o_2	o_3	i_1	i_2	i_3	o_1	o_2	o_3
0	0	0	0	0	0	0	0	1
0	0	1	1	0	0	1	0	1
0	1	0	0	0	1	0	0	0
0	1	1	0	0	1	0	0	0
1	0	0	0	1	0	0	1	1
1	0	1	1	0	0	1	0	1
1	1	0	0	1	0	0	1	1
1	1	1	0	0	0	0	0	1

Table 4.3. The function table of $f_{\hat{E}_{\text{ex2}}}$ and subsequent application of the local operations.

Example 9. The other example (see Equation 4.5 and Figure 4.8) consists of a uniform mixture of two cyclic channels: of

$$c : (o_1, o_2, o_3) \mapsto (o_3, o_1, o_2)$$

and of

$$\bar{c} : (o_1, o_2, o_3) \mapsto (o_3 \oplus 1, o_1 \oplus 1, o_2 \oplus 1) .$$

Suppose all three parties apply the identity operation ($o_j = i_j$). In that case, the left channel of Figure 4.8 (channel c) has *two* fixed points:

$$\begin{aligned} (0,0,0) &= c(0,0,0) , \\ (1,1,1) &= c(1,1,1) . \end{aligned}$$

Then again, the right channel (channel $\bar{c}$) has *no* fixed point:

$$\forall (i_1, i_2, i_3) \in \mathcal{I} : (i_1, i_2, i_3) \neq \bar{c}(i_1, i_2, i_3) .$$

Thus, in *average*, the number of fixed points is $1/2 \cdot 2 + 1/2 \cdot 0 = 1$. If, to look at a second case, party S_3 applies the bit-flip operation instead, then

$$\forall (i_1, i_2, i_3) \in \mathcal{I} : (i_1, i_2, i_3) \neq c(i_1, i_2, i_3 \oplus 1),$$

yet

$$\begin{aligned}(0,1,1) &= \bar{c}(0,1,1 \oplus 1),\\ (1,0,0) &= \bar{c}(1,0,0 \oplus 1).\end{aligned}$$

So, again, the number of fixed points on average is 1.

4.6 Non-causal correlations

By now, we have all tools needed to discuss *non-causal* correlations (see Definition 8). In the settings with one or two parties (see Examples 4 and 5), all correlations achievable are *causal*. The reason for this is that the environments in these cases represent either a state or a channel in one direction. This has been proven for arbitrary dimensions in the work by Oreshkov *et al.* [194] and in the work by Costa [85]. One of the main contributions of this thesis is that for three or more parties, there exist logically consistent environments that lead to *non-causal* correlations.

Assume a three-parties setup with the parties S_1, S_2, S_3. One way to prove that an environment produces *non-causal* correlations is by showing a *violation* of a causal inequality. These inequalities have been introduced in Section 3.2.

4.6.1 Probabilistic non-causal correlations

The non-causal correlations that are established here are *probabilistic*. Consider the following game:

Game 1. The three parties

$$\begin{aligned}S_1 &= (A_1 \equiv (A, M), X_1 \equiv X, L_1),\\ S_2 &= (A_2 \equiv (B, M), X_2 \equiv Y, L_2),\\ S_3 &= (A_3 \equiv (C, M), X_3 \equiv Z, L_3),\end{aligned}$$

have binary random variables A, B, C, X, Y, Z and M is a *shared* trit with $\mathcal{M} = \{1,2,3\}$. Assume that all free random variables are *uniformly* distributed. The goal of the game these three parties play is to maximize

$$\begin{aligned}p_{\text{succ}} := \frac{1}{3} (&\Pr(X = B \oplus C \mid M = 1)\\ &+ \Pr(Y = A \oplus C \mid M = 2)\\ &+ \Pr(Z = A \oplus B \mid M = 3)),\end{aligned}$$

which states that, depending on the value of M, one party should receives the parity of the inputs of the other two parties.

The maximal value this probability expression can take in a *causal* scenario is

$$p_{\text{succ}}^{\text{prob}} \leq \frac{5}{6},$$

which is a *causal inequality* (see Section 3.2). The reasoning here is that in a causal scenario, *at least one party is in the causal past of all others*. Without loss of generality, assume that $S_1 \preceq_{\text{ST}} S_2$ and that $S_1 \preceq_{\text{ST}} S_3$. If $M = 1$, then the guess X of S_1 is correct with half probability:

$$p_{\text{succ}}^{\text{prob}} \leq \frac{1}{3}\left(\frac{1}{2}+1+1\right) = \frac{5}{6}.$$

The same holds where any other party is in the causal past of the remaining two.

The environment $\hat{E}_{\text{ex1}}$ (shown in Figure 4.8 and in Equations (4.5) and (4.6)) with appropriate local operations *violates* this inequality up to the algebraic maximum. This violation is achieved with the following choice of local operations:

$$L_1 = P_{X,O_1|A,I_1,M}(x,o_1,a,i_1,m) = \begin{cases} \delta_{x,i_1}\delta_{o_1,0} & \text{if } m=1, \\ \delta_{x,0}\delta_{o_1,i_1\oplus a} & \text{if } m=2, \\ \delta_{x,0}\delta_{o_1,a} & \text{if } m=3, \end{cases}$$

$$L_2 = P_{Y,O_2|B,I_2,M}(y,o_2,b,i_2,m) = \begin{cases} \delta_{y,i_2}\delta_{o_2,0} & \text{if } m=2, \\ \delta_{y,0}\delta_{o_2,i_2\oplus a} & \text{if } m=3, \\ \delta_{y,0}\delta_{o_2,b} & \text{if } m=1, \end{cases}$$

$$L_3 = P_{Z,O_3|C,I_3,M}(z,o_3,c,i_3,m) = \begin{cases} \delta_{z,i_3}\delta_{o_3,0} & \text{if } m=3, \\ \delta_{z,0}\delta_{o_3,i_3\oplus a} & \text{if } m=1, \\ \delta_{z,0}\delta_{o_3,c} & \text{if } m=2. \end{cases}$$

That is, party S_M uses the input I_M from the environment as her guess to win the game. The other parties, *i.e.*, S_j with $j \neq M$, set, without loss of generality, there non-free random variable to 0. Party $S_{(M+1 \mod 3)+1}$ adds her free variable to the input from the environment and uses that as the output to the environment. Finally, party $S_{(M \mod 3)+1}$ simply uses the free variable as output to the environment. By this, the success probability for Game 1 is 1. The case where $M = 1$ is shown in Figure 4.13. If $M = 1$, then the correlations that the parties establish are such that $S_2 \preceq S_1$ and $S_3 \preceq S_1$. For $M = 2$, we get $S_1 \preceq S_2$ and $S_3 \preceq S_2$. Finally, for $M = 3$, we get $S_1 \preceq S_3$ and $S_2 \preceq S_3$. Thus, the causal relations among the parties *depend* on the choice of the local operations they apply.

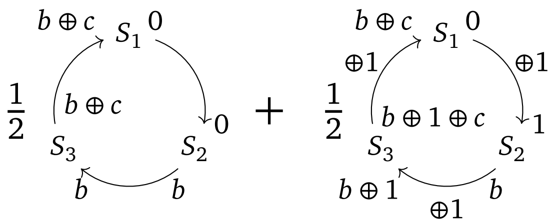


Figure 4.13. For $M = 1$, party S_1 outputs 0 to the environment, party S_2 outputs b to the environment, and party S_3 adds c to the input from the environment to produce the output. Via both circular channels, party S_1 receives the correct input $b \oplus c$ from the environment (the two bit flips of the right channel cancel).

4.6.2 Deterministic non-causal correlations

Deterministic environments that are logically consistent and can be used to establish non-causal correlations exist as well. We already found such an example as an extremal point of the deterministic-extrema polytope (see Section 4.4.2). Here, we show that the environment depicted in Figure 4.11 and Equations (4.7) and (4.8) indeed leads to *non-causal* correlations. For this we introduce the following game:

Game 2. Consider the three parties

$$\begin{aligned} S_1 &= (A_1 \equiv A, X_1 \equiv X, L_1), \\ S_2 &= (A_2 \equiv B, X_2 \equiv Y, L_2), \\ S_3 &= (A_3 \equiv C, X_3 \equiv Z, L_3), \end{aligned}$$

where all random variables are binary and the free random variables are *uniformly* distributed. The game is that the three parties wish to maximize the probability expression

$$\begin{aligned} p_{\text{succ}}^{\text{det}} := \frac{1}{2} (&\Pr(X = C \wedge Y = A \wedge Z = B \mid \text{maj}(A, B, C) = 0) \\ &+ \Pr(X = B \oplus 1 \wedge Y = C \oplus 1 \wedge Z = A \oplus 1 \mid \text{maj}(A, B, C) = 1)) , \end{aligned} \tag{4.9}$$

where the function $\text{maj}(A, B, C)$ takes three bits and outputs the majority 0 or 1 of the bits. Thus, if the majority of the free variables is 0, then the parties aim to establish a clock-wise identity channel. Otherwise, they aim to establish a counter-clockwise bit-flip channel.

In a causal scenario, the highest value for the success probability is $3/4$. The reason for this is that a causal scenario forces at least one party to be in the causal past of the other two parties, *e.g.*, $S_1 \preceq_{\text{ST}} S_2$ and $S_1 \preceq_{\text{ST}} S_3$. In that case, S_1 has to guess $\neg B \wedge C$, which, for uniformly distributed B, C, is 0 with probability $3/4$. Thus, party S_1 has to bet on 0 to maximize the probability expression; a bet she loses with probability $1/4$.

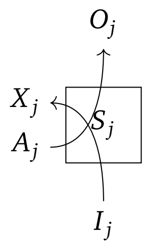


Figure 4.14. The local operation of party S_j with $j \in \{1,2,3\}$ is to simply forward the free variable as the input to the environment, and to use the state obtained from the environment as the guess.

From this follows that the overall probability for all parties to guess correctly is upper bounded by 3/4.

The environment from Figure 4.11, however, can be used to win the game *perfectly* — this environment implements precisely the relations the parties wish to achieve. For that purpose, we simply choose the trivial operations as the local operations of the parties (see Figure 4.14):

$$L_1 = P_{X,O_1|A,I_1}(x, o_1, a, i_1) = \delta_{x,i_1} \delta_{o_1,a} ,$$
$$L_2 = P_{Y,O_2|B,I_2}(y, o_2, b, i_2) = \delta_{y,i_2} \delta_{o_2,b} ,$$
$$L_3 = P_{Z,O_3|C,I_3}(z, o_3, c, i_3) = \delta_{z,i_3} \delta_{o_3,c} .$$

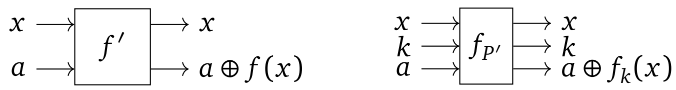
The condition (4.9) corresponds to the following set of equations:

$$X = \neg B \wedge C ,$$
$$Y = \neg C \wedge A ,$$
$$Z = \neg A \wedge B .$$

The causal relations among the parties clearly depends on the free variables of the parties, *i.e.*, if party S_3 uses $C = 0$, then $S_1 \preceq S_2$, otherwise, $S_2 \preceq S_1$. From this we see that a party can select the signaling direction between the remaining two parties.

4.7 Reversible environments

The above environments are processes, or channels, that produce the inputs to the parties from their outputs. Here, we pose the question whether such processes can be made *reversible*. A reversible process is necessarily a function; and a function e is reversible if and only if its *inverse* e^{-1} exists as well ($e^{-1}(e(x)) = x$). In general, any function can be embedded into a reversible one — this is usually done by embedding it into a function with a higher-dimensional domain and range. However, it is not immediately clear whether this can be done for processes. The caveat here is that the



(a) (b)

Figure 4.15. (a) Reversible embedding of a function f. (b) Reversible embedding of a probabilistic operation (channel) $P_{Y|X}(\cdot,x) = \sum_k p_k f_k(x)$.

larger process (in which some logically consistent process is embedded) needs to be logically consistent as well.

A function

$$f : \mathcal{X} \to \mathcal{Y}, \\ x \mapsto y,$$

with finite sets $\mathcal{X} = \{0,1,\dots,n_x\}, \mathcal{Y} = \{0,1,\dots,n_y\}$, can be embedded into a reversible function f' in the following way (see Figure 4.15a):

$$f' : \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}, \\ (x,a) \mapsto (x, a \oplus f(x)),$$

where $\oplus$ is the addition modulo $|\mathcal{Y}|$. Then, f is retrieved from f' whenever the second input to f' is 0: $f'(x,0) = (x, f(x))$. The same holds for probabilistic operations by the technique of derandomization. There, a channel $P_{Y|X}$ has first to be decomposed into a convex mixture of deterministic channels (functions)

$$P_{Y|X}(\cdot,x) = \sum_k p_k f_k(x),$$

where p_k are probabilities and f_k are functions from $\mathcal{X}$ to $\mathcal{Y}$. The reversible function $f_{P'}$ is defined as

$$f_{P'} : \mathcal{X} \times \mathcal{K} \times \mathcal{Y} \to \mathcal{X} \times \mathcal{K} \times \mathcal{Y}, \\ (x,k,a) \mapsto (x,k,a \oplus f_k(x)),$$

where $\mathcal{K}$ is the set that enumerates the functions used in the decomposition. For the probabilistic input $P_K(k) = p_k$, this construction recovers the distribution $P_{Y|X}(\cdot,x)$. Then, a *reversible* channel can be implemented as shown in Figure 4.15b.

Some, but not all, environments studied here can be embedded into a reversible process as well. In that context, we present three results. First, *all* environments from the *deterministic-extrema polytope* (from set $\mathcal{D}$) can be embedded into a reversible process. Second, *no* environment from *outside* of that polytope can be made reversible. And finally, to make an environment reversible (if possible), one needs additional parties; a so-called *source* and a so-called *sink*.

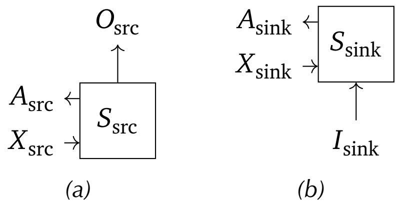


Figure 4.16. (a) The source S_{src} is a party that does not receive any system from the environment. (b) The sink S_{sink} is a party that does not provide any system to the environment.

4.7.1 Reversible environments from the deterministic-extrema polytope

We constructively show how to embed processes from the set $\mathcal{D}$, and prove that every other process cannot be made reversible. In the spirit of Figures 4.15a and 4.15b, we add two parties: a source, labelled by S_{src}, and a sink, labelled by S_{sink}. The source has a trivial input and produces an output to the environment (see Figure 4.16a). Her operation is described by the conditional probability distribution $P_{X_{src},O_{src}|A_{src}}$. Symmetrically, the sink has a trivial output and obtains a random variable from the environment (see Figure 4.16b). Here, the local operation is $P_{X_{sink}|A_{sink},I_{sink}}$.

Theorem 7 (Reversible extremal ponits from the deterministic-extrema polytope)**.** *Every environment that corresponds to an extremal point of the deterministic-extrema polytope (from the set $\mathcal{D}_{ext}$) can be embedded into a logically consistent environment that is* reversible.

Proof. Let $\hat{E} \in \mathcal{D}_{ext}$ be an extremal point of the deterministic-extrema polytope. Therefore, f_E represents a function

$$\begin{aligned} f_E : \mathcal{O} &\to \mathcal{I}\,, \\ o &\mapsto i\,. \end{aligned}$$

The reversible and logically consistent environment into which $\hat{E}$ is embedded is defined as

$$\begin{aligned} e' : \mathcal{O} \times \mathcal{I}_{src} &\to \mathcal{I} \times \mathcal{O}_{sink}\,, \\ (o, r) &\mapsto (r \oplus f_E(o), o)\,, \end{aligned}$$

where the source and the sink are incorporated. Here, the operation $\oplus$ is an addition modulo $|\mathcal{I}_j|$ that is applied locally to every party S_j. The output space of the source is $\mathcal{O}_{src} = \mathcal{I}$, and the input space of the sink is $\mathcal{I}_{sink} = \mathcal{O}$. This setup is visualized in Figure 4.17. As in the considerations above, we simply ignore the input and output spaces $\mathcal{A}_{src}, \mathcal{A}_{sink}, \mathcal{X}_{src}, \mathcal{X}_{sink}$ of the source and sink. We have to show two properties

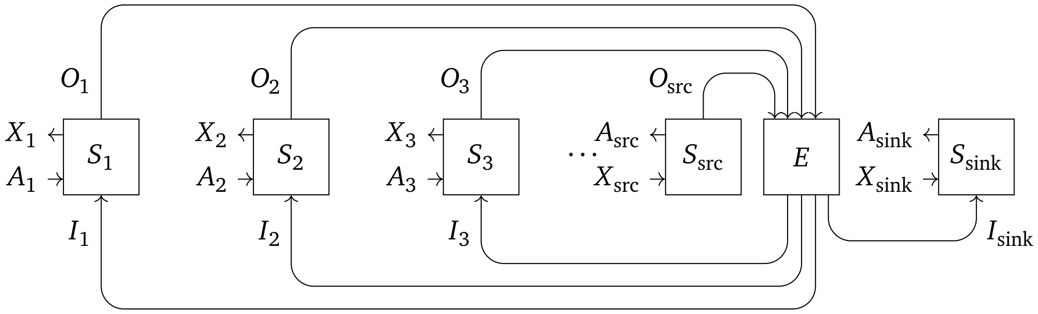


Figure 4.17. A reversible embedding of an environment $\hat{E}$ from the deterministic-extrema polytope (set $\mathcal{D}_{\text{ext}}$).

of e'. The first is that it indeed embeds f_E, and the second one is that it is logically consistent. For the latter property, we also need to look at *all* operations the source and the sink can apply.

Formally, the first property is

$$\exists r \in \mathcal{O}_{\text{src}}, \forall o \in \mathcal{O} : e'(o,r)_{\mathcal{O}} = f_E(o)\,.$$

By the subscript $\mathcal{O}$ we mean that we look only at the space $\mathcal{O}$ after e' has been applied. Indeed, by construction, this property is fulfilled for $r = 0$.

The second property is

$$\forall r \in \mathcal{O}_{\text{src}}, \forall L \in \mathcal{L}, \exists!(i,s) \in \mathcal{I} \times \mathcal{I}_{\text{sink}} : (i,s) = e'(f_L(i), r)\,.$$

Note that here we applied the same notation as above again: If an index is dropped, then we refer to the collection. We rewrite the above equation with $e' \equiv (e'_1, e'_2)$:

$$\forall r \in \mathcal{O}_{\text{src}}, \forall L \in \mathcal{L}, \exists!(i,s) \in \mathcal{I} \times \mathcal{I}_{\text{sink}} : i = e'_1(f_L(i), r) \wedge s = e'_2(f_L(i), r)\,.$$

The latter part is trivially satisfied: e'_2 is a function and thus always outputs one value. The former part follows from the assumption that f_E is logically consistent. First, let us rewrite e'_1:

$$i = e'_1(f_L(i), r) = r \oplus f_E(f_L(i))\,.$$

The addition modulo $|\mathcal{I}_j|$ can now be carried into the local operation of party S_j. Since f_E has a unique fixed point for every choice of local operation, it also has a unique fixed point for this modified local operation. By this, e' is logically consistent as well. □

The above theorem is designed for the *extremal* points of the deterministic-extrema polytope. However, it is easy to see that the same technique can be applied to show that every point *inside* that polytope, *i.e.*, every environment from the set $\mathcal{D}$, can be embedded into a reversible process.

Corollary 1 (Reversibility of environments from within the deterministic-extrema polytope)**.** *Every environment that corresponds to a point* inside *the deterministic-extrema polytope can be embedded into a logically consistent environment that is* reversible*.*

This is obtained by first decomposing some environment $\hat{E} \in \mathcal{D}$ into a convex combination of the extremal points of the set $\mathcal{D}_{\text{ext}}$. Thereafter, every extremal point is made reversible by the theorem above. And finally, one applies the derandomization technique (see Figure 4.15b). Here, however, the source has to produce an additional system to the environment which selects the extremal point to be implemented. The sink, then again, needs an additional input from the environment where she receives this selection.

4.7.2 Environments from outside of the deterministic-extrema polytope cannot be made reversible

Every logically consistent process that lies *outside* of the polytope in consideration, *i.e.*, an environment that is *not* in the set $\mathcal{D}$, *cannot* be embedded into a logically consistent environment that is reversible.

Theorem 8 (Impossiblity of reversible pure mixtures)**.** *Any logically consistent environment that is* not *in the set $\mathcal{D}$ cannot be made reversible.*

Proof. Take any environment $\hat{E} \notin \mathcal{D}$. Such an environment can be decomposed as

$$\hat{E} = \sum_i p_i \hat{E}_i \,,$$

where p_i are probabilities and $\hat{E}_i$ are *deterministic* environments, where some are *logically inconsistent*. Note that such a decomposition might not be unique. However, this proof works for *any* decomposition into deterministic environments. Since $\hat{E} \notin \mathcal{D}$, some of the $\hat{E}_i$ are logically *in*consistent. In the most general setting, $\hat{E}$ is embedded into a larger process with a source and a sink. Because $\hat{E}$ is probabilistic and the embedding should be deterministic (reversible), the source has an additional output that is fed to the larger environment which selects the deterministic process. Thus, the source could simply select the process from the decomposition that is logically *in*consistent, and by this create a contradiction. □

4.7.3 Necessity of some source and some sink

Finally, we show that if an environment can be made reversible, then only with additional parties.

Theorem 9 (Necessity of a sournce and a sink to make environments reversible)**.** *If every party obtains an input and provides an output to the environment, then every reversible function $f : \mathcal{O} \to \mathcal{I}$ is logically* inconsistent*.*

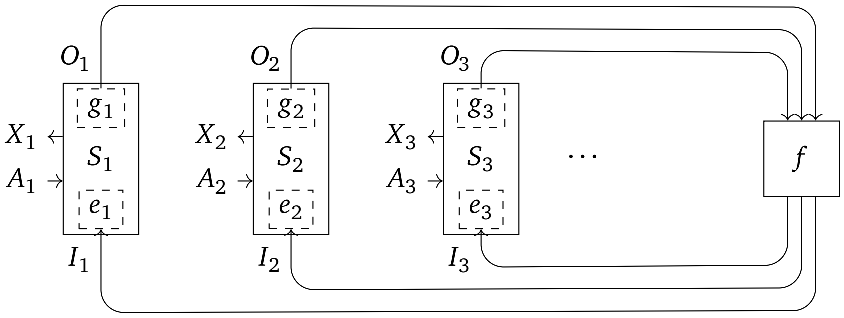


Figure 4.18. The function f is assumed to be reversible and, towards a contradiction, logically consistent. Every party S_j applies some pre- (e_j) and post processing (g_j) as part of their local operation.

Proof. Suppose $f : \mathcal{O} \to \mathcal{I}$ is reversible, and, (towards a contradiction) logically consistent. Since f is reversible, we have $\mathcal{O} \equiv \mathcal{I}$. Furthermore, suppose without loss of generality that $\mathcal{O}$ is isomorphic to $\{0,1\}^n$ for some n. Thus, there exists an encoding from $\mathcal{O}$ to $\{0,1\}^n$ and back. If the cardinality of $\mathcal{O}$ is not a power of 2, then one could just take the larger power of 2 and conduct the following proof for the subspace isomorphic to $\mathcal{O}$. Because f is logically consistent, there exists a *unique* fixed point for every choice of local operations $L \in \mathcal{L}$ (see Theorem 5). Now, suppose that every party applies some pre- and post-processing e_j and g_j that consist of bit-wise identity or bit-flip operations (see Figure 4.18), such that the fixed point of $f' = e \circ f \circ g$ is 0^n. The function f' is reversible (because f, g, and e are reversible) and logically consistent. The latter follows because e and g are just local operations, and f is supposed to be consistent for every choice of local operations. Let $\ell = (\ell_1, \ell_2, \dots)$ be the local operations of the parties *after* the preprocessing and *before* the postprocessing. There exist 2^n different ℓ that consist of the identity or the bit-flip operations only. For different ℓ, the fixed point of $f' \circ \ell$ is different as well: If $\ell \neq \tilde{\ell}$, then $\tilde{\ell} = \ell \circ \ell_\Delta$, and therefore, the fixed point y of $f' \circ \ell$ is not a fixed point of $f' \circ \tilde{\ell}$:

$$\begin{aligned} f' \circ \tilde{\ell}(y) &= f' \circ \ell \circ \ell_\Delta(y), \qquad \text{but} \\ \ell_\Delta(y) &\neq y\,. \end{aligned}$$

This means that for every $y \in \{0,1,\}^n$, there exists an ℓ such that y is the fixed point. In particular, there exists an ℓ' where the fixed point is 1^n. This local operation ℓ', however, is different from the identity operation (there, the fixed point is 0^n). Furthermore, it is also different from the bit-wise bit-flip on every position. In that case, $\ell'(1^n) = 0^n$ would hold, however, $f'(0^n) = 0^n$ holds as well. This means, ℓ' consists of a strictly positive number of identity, and strictly positive number of bit-flip operations. Now, define the following local operation m. The operation m is equal to ℓ' up to every position where ℓ' implements a bit-flip. On these positions, m implements the constant 0 as opposed to a bit-flip. Now, we have $m(1^n) = \ell'(1^n)$, which means that 1^n is the fixed point of $f' \circ m$.

However, and this is where the contradictions comes in, 0^n is a fixed point of $f' \circ m$ as well:

$$f' \circ m(0^n) = f'(0^n) = 0^n \,.$$

Thus, $f' \circ m$ has at least *two* fixed points, and by this contradicts our initial assumption. □

4.7.4 Example of a reversible non-causal environment

Let us apply Theorem 7 to the environment of Equation (4.8). By this, we obtain a *reversible* environment with which Game 2 can be won. The environment (4.8) is designed for three parties and binary random variables. Since (4.8) is a *deterministic* environment, we can write it as a function (see also Example 8). Now, the parties S_1, S_2, S_3 input states i_1, i_2, i_3 are the result of some function e applied to their output states o_1, o_2, o_3. The function e, which is equivalent to the environment (4.8) is

$$\begin{aligned} e : \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3 &\to \mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3 \,, \\ (o_1, o_2, o_3) &\mapsto (\neg o_2 \wedge o_3, \neg o_3 \wedge o_1, \neg o_1 \wedge o_2) \,. \end{aligned}$$

To make e reversible, we need to add two parties: a source S_{src} and a sink S_{sink}. The output state space of the source is $\mathcal{O}_{\text{src}} \equiv \mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3$, and the input state space of the sink is $\mathcal{I}_{\text{sink}} \equiv \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3$. The reversible function is

$$\begin{aligned} e_{\text{rev}} : \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3 \times \mathcal{O}_{\text{src}} &\to \mathcal{I}_1 \times \mathcal{I}_2 \times \mathcal{I}_3 \times \mathcal{I}_{\text{sink}} \,, \\ (o_1, o_2, o_3, r_1, r_2, r_3) &\mapsto (\neg o_2 \wedge o_3 \oplus r_1, \neg o_3 \wedge o_1 \oplus r_2, \neg o_1 \wedge o_2 \oplus r_3, o_1, o_2, o_3) \,. \end{aligned} \tag{4.10}$$

In the same spirit, every other process from the set $\mathcal{D}$ can be embedded into a reversible one. The sink, in such an embedding, receives the output states of the parties, and the source provides some states that are added after the original function has been applied.

4.8 Quantum correlations without causal order

The framework developed in this chapter is a special case (the classical limit) of its quantum variant: the process-matrix framework [194]. The process-matrix framework — which was developed *prior* to this framework — uses the same assumptions as listed in the Section 4.1, with the sole difference that the underlying states are *quantum* states as opposed to classical states. The quantum version allows for correlations not obtainable in the classical limit. Most strikingly, and in stark contrast to the classical special case, the process-matrix framework can establish *non-causal* correlations in a *two*-party setup [85, 194] (see also Reference [159]). An example of a two-party causal inequality is Inequality (3.1). That inequality cannot be violated in the framework developed here. In the quantum framework, then again, a violation is possible, and the maximal violation of that inequality (under some restrictions) is $(2+\sqrt{2})/4$ [57]. This violation coïncides numerically with Tsirelson's bound [73]: the maximal *quantum* violation of a specific Bell inequality. It is interesting to see that here the same algebraic number pops up in to different contexts: non-local and non-causal correlations.

In the process-matrix framework, the *process matrix* is the analogous mathematical object to what we called "environment," or "classical process." Interestingly, some process-matrices exist that do *not* violate any causal inequality, but which are nevertheless *non separable* [115]. A process matrix is non separable if it cannot be written as a convex combination of process matrices with *fixed* causal structures. To detect this non separability, so-called *causal witnesses* [16, 52, 114, 115, 173] have been developed — analogous to the entanglement wittinesses. The process-matrix framework has also been extended to *continuous variables* [125].

Even though the quantum formalism can be seen as a generalization of our formalism, it was unclear whether unitary quantum processes that violate causal inequalities exist. The reason for why this was unclear initially is that in the quantum setup, an infinite number of operations are possible, also for a finite-dimensional state space. This means, for a *quantum* process to be logically consistent, the derived probabilities must be well-defined for *every* operation (an infinity) of all parties. Just as in the classical formalism, the physical realizability of non-causal correlations is highly debatable. This led Araújo *et al.* [17] to formulate a *purification postulate*: "processes are physical only if they are purifiable." *Purifiable* means that a process can be embedded into a larger process that is *pure* — a term often used for *unitary* (and hence, reversible) quantum channels. Since no violation of any causal inequality has been observed, it is tempting to conjecture that all process matrices that lead to *violations* of causal inequalities *cannot* be purified. However, as is shown here, this is not true; we can transform the *reversible* classical process from Section 4.7.4 to a *pure quantum process*. As a side remark, note the close connection of this discussion to the reversibility principle in the following study of closed time-like curves.

We recap the process-matrix framework. After that, we present the unitary analogue of the reversible classical process from Section 4.7.4.

4.8.1 Framework

Since the underlying states are quantum (as opposed to classical ones), a party is defined by a *quantum channel* from her inputs to her outputs. In accordance with Definition 6, a party S_j has a free random variable A_j and a non-free random variable X_j. These variables are *classical*: A_j represents the *measurement setting*, and X_j represents the measurement outcome of party S_j.[2] The systems that a party S_j receives from and returns to the environment are, in contrast to the classical setting and result, *quantum*

[2]More precisely, these classical variables are *related* to the setting and outcome, as some processing could happen in between, *e.g.*, the measurement operator applied could be independent of A_j.

states. We label the input system of party S_j by I_j, and the output system is O_j. The state space of the input to party S_j is $\mathcal{S}(\mathcal{H}_{I_j})$. For the sake of a simpler presentation, we denote this state space by $\mathcal{I}_j$. The same is done for the output state space of a party: The set $\mathcal{S}(\mathcal{H}_{O_j})$ is denoted by $\mathcal{O}_j$. Thus, the local operation L_j is a quantum channel from $\mathcal{I}_j$ to $\mathcal{O}_j$. In the most general setting, such a channel is a completely-positive and trace-preserving (CPTP) map. Such a CPTP map can be split up into a collection of completely-positive (CP) maps. A CP map can be understood as a measurement, where the probability that a particular map is applied equals to the trace of the resulting state.

We represent the local operation L_j of a party S_j by its Choi map $M_j^{x_j,a_j}$ (see Appendix), which depends on the measurement setting a_j and on the measurement result x_j. If we sum over all measurement results x_j, then we obtain

$$\sum_{x_j} M_j^{x_j,a_j} = M_j^{a_j} \,,$$

which is a CPTP map. There is a slight difference in notation of the presentation of the process-matrix framework here when compared to the initial article by Oreshkov, Costa, and Brukner [194]: In the Oreshkov *et al.* [194] notation, the Choi maps are additionally transposed.

If all local operations of the parties are given, then the *most general expression* satisfying all assumptions (see Section 4.1) that gives the probability distribution $P_{X|A}$ is:

$$P_{X_1,X_2,\dots|A_1,A_2,\dots}(x_1,x_2,\dots,a_1,a_2,\dots) = \mathrm{Tr}\left(\left(M_1^{x_1,a_1} \otimes M_2^{x_2,a_2} \otimes \dots\right) W\right) .$$

The mathematical object W is a matrix in $\mathcal{L}(\mathcal{H}_{I_1} \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{I_2} \otimes \mathcal{H}_{O_2} \otimes \dots)$. The condition that all probabilities are non-negative and sum to unity produces a set of restrictions for W. The non-negativity condition is

$$W \geq 0 \,, \tag{4.11}$$

and the unit-probability condition is

$$\forall M_1, M_2, \dots : \mathrm{Tr}((M_1 \otimes M_2 \otimes \dots) W) = 1 \,, \tag{4.12}$$

where M_j are CPTP maps represented as as Choi maps. Theses conditions imply

$$\mathrm{Tr}_{I_1,I_2,\dots} W = \mathbb{1} \,.$$

Thus, W can be interpreted as a CPTP map from $\mathcal{O}_1, \mathcal{O}_2, \dots$ to the spaces $\mathcal{I}_1, \mathcal{I}_2, \dots$ in the Choi-Jamiołkowski (CJ) representation (see Figure 4.19). Such a W is called *process matrix* and corresponds to the quantum analog of what we called *environment* or *classical process* (see Definition 12). The conditions on the process matrix imply that if a process matrix is decomposed into a weighted sum of Pauli matrices, then for every summand appearing in the decomposition, the following must hold (see References [193, 194]): There exists at *least* one party with the identity operator in her output space and an operator *different* from identity on her input space. The interpretation of this is that at least one party *receives* some state *without* sending a state. A process matrix that satisfies these conditions is called *valid*.

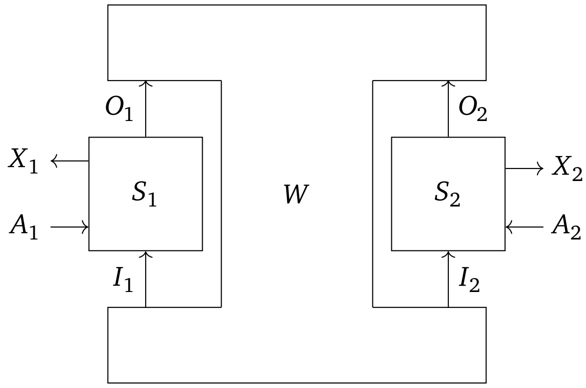


Figure 4.19. A process matrix W for two parties S_1 and S_2.

4.8.2 Reversible non-causal process matrix

As said before, a natural conjecture could be that process matrices leading to a violation of causal inequalities cannot be embedded into a larger unitary process that leads to well-defined probabilities as well. The motivation for posing such a conjecture is that violations of causal inequalities seem unphysical — or, in other words, their physical existence is doubted. Here, we show that this conjecture is false, and give an explicit process matrix which represents a unitary process and leads to well-defined *non-causal* probabilities. The process matrix W^U is constructed from the reversible classical process from Section 4.7.4 and is a rank 1 Choi map, *i.e.*, it represents a *unitary* transformation.

Assume three parties S_1, S_2, S_3, a source S_{src}, and a sink S_{sink}. Let us define a unitary transformation U from the spaces $\mathcal{O} \times \mathcal{O}_{\mathrm{src}}$ to the spaces $\mathcal{I} \times \mathcal{I}_{\mathrm{sink}}$, that implements the function (4.10) in some fixed basis:

$$U = \sum_{\substack{o_1,o_2,o_3\\r_1,r_2,r_3}} |\neg o_2 \wedge o_3 \oplus r_1, \neg o_3 \wedge o_1 \oplus r_2, \neg o_1 \wedge o_2 \oplus r_3, o_1, o_2, o_3\rangle\langle o_1, o_2, o_3, r_1, r_2, r_3| \,.$$

In the CJ representation, this unitary is the following process matrix

$$\begin{aligned} W^U &= |U\rangle\rangle\langle\langle U|\,, \qquad \text{with} \qquad (4.13)\\ |U\rangle\rangle &= \sum_{\substack{o_1,o_2,o_3\\r_1,r_2,r_3}} |\neg o_2 \wedge o_3 \oplus r_1, \neg o_3 \wedge o_1 \oplus r_2, \neg o_1 \wedge o_2 \oplus r_3, o_1, o_2, o_3\rangle \times \\ &\qquad |o_1, o_2, o_3, r_1, r_2, r_3\rangle \,. \end{aligned}$$

Clearly, the process matrix W^U implements the function e_{rev} in the computational basis. This means, it leads to violations of a causal inequality. Furthermore, it can easily be checked that U is a unitary and that W^U is the corresponding operator in the CJ representation. Thus, it remains to show that W^U is *valid*.

A process matrix W^U is valid if and only if the following conditions hold. The matrix W^U must be a CPTP map from $\mathcal{O}, \mathcal{O}_{\text{src}}$ to $\mathcal{I}, \mathcal{I}_{\text{sink}}$ in the CJ from

$$W^U \geq 0 \,,$$
$$Tr_{I,I_{\text{sink}}} W = \mathbb{1} \,,$$

and, in the Pauli decomposition of W^U, every summand must contain *at least* one party who has the identity operator on the output and some different operator on the input. Since party S_{sink} has *no* output space, S_{sink} always receives a state — except in those terms of the Pauli decomposition where I_{sink} is the identity operator. In this latter case, the sink does not receive any state. Thus, all terms in the Pauli decomposition that are different from identity on the input space of S_{sink} do *not* hinder the process matrix to be valid. By this, we can simplify our analysis to the reduced matrix $W'^U = Tr_{I_{\text{sink}}} W^U$. In this reduced matrix, all summands of a Pauli decomposition where I_{sink} is the identity operator vanish into a multiplicative number. We define the operator ${}_QX$ as in Ref. [16], to cover the remaining cases:

$${}_QX = \frac{\mathbb{1}_Q}{d_Q} \otimes Tr_Q X \,,$$

where d_Q is the dimension of the system Q and id_Q is the d_Q-dimensional identity operator.

If two parties of S_1, S_2, S_3 are not involved in a summand of a Pauli decomposition (their spaces contain the identity operator), then the third party must have *identity on her output*. This is expressed by the following set of equations:

$$\begin{aligned}
{}_{I_1,O_1,I_2,O_2}W'^U &= {}_{I_1,O_1,I_2,O_2,O_3}W'^U \,, \\
{}_{I_1,O_1,I_3,O_3}W'^U &= {}_{I_1,O_1,I_3,O_3,O_2}W'^U \,, \\
{}_{I_2,O_2,I_3,O_3}W'^U &= {}_{I_2,O_2,I_3,O_3,O_1}W'^U \,.
\end{aligned}$$

If one party of S_1, S_2, S_3 is not involved in a summand of a Pauli decomposition, then one of the remaining tow parties must have the identity operator on the output:

$$\begin{aligned}
{}_{I_1,O_1}W'^U &= {}_{I_1,O_1,O_2}W'^U + {}_{I_1,O_1,O_3}W'^U - {}_{I_1,O_1,O_2,O_3}W'^U \,, \\
{}_{I_2,O_2}W'^U &= {}_{I_2,O_2,O_1}W'^U + {}_{I_2,O_2,O_3}W'^U - {}_{I_2,O_2,O_1,O_3}W'^U \,, \\
{}_{I_3,O_3}W'^U &= {}_{I_3,O_3,O_1}W'^U + {}_{I_3,O_3,O_2}W'^U - {}_{I_3,O_3,O_1,O_2}W'^U \,.
\end{aligned}$$

In the last case, at least one party of S_1, S_2, S_3 has identity on the output:

$$W'^U = {}_{O_1}W'^U + {}_{O_2}W'^U + {}_{O_3}W'^U - {}_{O_1,O_2}W'^U - {}_{O_1,O_3}W'^U - {}_{O_2,O_3}W'^U + {}_{O_1,O_2,O_3}W'^U \,.$$

All these conditions are satisfied by the *reversible* and logically consistent process matrix W^U (see Equation (4.13)). Furthermore, W^U leads to violations of causal inequalities.

4.8.3 The classical framework arises as limit of the quantum framework

We quickly show that the classical framework for correlations without causal order is indeed recovered if one takes the classical limit of the process-matrix framework. The mathematical objects in the process-matrix framework are local operations and the process matrix, which all can be interpreted as Choi maps. In the classical limit, these Choi maps become *diagonal* in the computational basis.

Theorem 10 (The classical limit of the process-matrix framework is the classical framework for correlations without causal order)**.** *In the classical limit, the conditions expressed in Equation* (4.11) *and Equation* (4.12) *equal to conditions of Equation* (4.4) *and Equation* (4.3).

Proof. We first show a bijection of the quantum object in the classical limit (diagonal Choi maps) to the stochastic operations used in the classical framework. Suppose $\hat{E}$ is a stochastic matrix that describes the environment. Then, the process matrix W_E is defined as

$$W_E = \sum_k |k\rangle\langle k|_O \otimes d\left(\hat{E}|k\rangle \sum_\ell \langle \ell|_I\right),$$

where $|k\rangle$ and $|\ell\rangle$ have the same dimension as $\hat{E}$. The function d is the function that takes a matrix and cancels all off-diagonal terms, *i.e.*,

$$d(\rho) = \sum_m |m\rangle\langle m|\rho|m\rangle\langle m|\,.$$

By this, W_E is rewritten as

$$W_E = \sum_k |k\rangle\langle k|_O \otimes \sum_m |m\rangle\langle m|\hat{E}|k\rangle\langle m|_I\,.$$

This Choi map W_E acts in the same way as $\hat{E}$: Some state $|k\rangle$ is mapped to $\hat{E}|k\rangle$. A local operation $\hat{L}'_j$ is transformed to a Choi map in a similar way:

$$M'_j = \sum_{k',m'} |m'\rangle\langle m'|\hat{L}'_j|k'\rangle\langle m'|_{O_j} \otimes |k'\rangle\langle k'|_{I_j}\,.$$

The reverse direction of the bijection is given by these definitions.

We explicitly show (in the restricted case of one party only) that the conditions coïncide. That condition (4.11) coïncides with (4.4) is easy to see. That condition (4.12) equals (4.3) is shown in the following calculation, where we assume that the process matrix W and the local operation M are diagonal in the computational basis.

$$\mathrm{Tr}(WM_1) = \sum_{i,j} \langle i,j|WM_1|i,j\rangle = \sum_{i,j} \langle i,j|W|i,j\rangle\langle i,j|M_1|i,j\rangle\,.$$

By substituting W with W_E and M_1 with M_1' we get

$$\sum_{i,j,m,k,m',k'} \langle i|k\rangle\langle k|i\rangle\langle j|m\rangle\langle m|\hat{E}|k\rangle\langle m|j\rangle\langle i|m'\rangle\langle m'|\hat{L}_1'|k'\rangle\langle m'|i\rangle\langle j|k'\rangle\langle k'|j\rangle$$
$$= \sum_{i,j}\langle j|\hat{E}|i\rangle\langle i|\hat{L}_1'|j\rangle$$
$$= \mathrm{Tr}\left(\hat{E}\hat{L}_1'\right),$$

which concludes the proof. The case of more than one party works analogously. □

4.9 Discussion

Classical correlations without causal order are obtained by manipulating random variables in a way where only local assumptions are enforced. Among those local assumptions, we employed the assumption of logical consistency, which ensures that we do not run into a logical contradiction. By studying these correlations, a world between the logically possible and the causal opens: The framework developed allows for correlations that cannot be simulated if we assume some background time. This can be interpreted in the following way: From the correlations we can *deduce* the causal relations between the parties. In that sense, the causal relations *emerge* from — and as we show, depend on — the local operations. Thus, in the spirit of Leibniz (see Section 2), causal relations cannot be put on an absolute ground.

We developed the framework and characterized the mathematical objects in different ways. An interesting characterization is the one in terms of fixed points: It gives an intuitive understanding of the condition of logical consistency. In its deterministic form, the fixed-point theorem says that some process is logically consistent if and only if the process combined with the local operations has a *unique* fixed point — and that for *any* choice of local operations. Hence, a consistent assignment of values to the variables must exist for the "circular map" (no grandfather antinomy), yet *not more than one* (no information antinomy) is allowed.

We presented some causal and some non-causal correlations explicitly. It was conjectured that all correlations obtainable with reversible dynamics are *causal*. The reason to place such a conjecture is that we do not know *how* causal inequalities are violated, and enforcing reversibility seems natural. However, by showing that a large class of non-causal correlations can be obtained in a reversible way, and that these correlations are also consistent with the quantum framework, we refuted this conjecture.

Chapter 5

Closed time-like curves

> It is common belief that the 'identifications' [...] along the time axis are forbidden b[y] the laws of physics and must be excluded. This is usually justified by causality considerations. [...] Here we note only that, in general, the closure of time-like curves is not linked with the causality principle in such a trivial manner. The clos[ure] of time curves does not necessarily imply a violation of causality, since the events along such a closed line may be all 'self-adjusted' — they all affect one another through the closed cycle and follow one another in a self-consistent way.
>
> Igor Dmitriyevich Novikov [190]

In the previous chapter, we discussed non-causal correlations. These correlations were obtained by dropping the often made assumption of a global causal order; we did *not assume* any *a priori* causal relations among the parties. Here, we look at these results from the point of view of relativity theory. In relativity theory, space-time is *dynamic* — the space-time relations depend on the velocity, acceleration, *etc.* —, yet object are embedded *within* that space-time structure. The latter becomes clear if one describes a universe in general relativity without any objects: space-time persists. These thoughts, at a first glance, suggest that non-causal correlations cannot be described, or do not arise, within relativity theory. Also Einstein [99] commented on his potential discovery that (at that time not yet finished) general relativity allows for CTC by: "Dies widerstrebt meinem physikalischen Gefühl aufs lebhafteste."[1] Furthermore, the *limitation on the speed of light*, and therefore of any signal, seems to *prohibit causal loops* from arising in relativity theory. As Einstein [98] put it, the possibility of superluminal signaling means "dass wir einen Übertragungsmechanismus für möglich halten müssten, bei dessen Benutzung die erziehlte Wirkung der Ursache vorangeht."[2] As a side remark, Einstein continues by writing that "dieses Resultat [...], meiner Ansicht nach, rein logisch genommen, keinen Widerspruch enthält."[3] In similar words, Tolman [236] writes: "Such a condition of affairs might not be a logical impossibility; nevertheless its extraordinary nature might incline us to believe that no causal impulse can travel with a velocity greater than that of light." However, if an agent observes an effect that *precedes* its cause, and if that agent has the ability of superluminal signaling, then we should worry about logical inconsistencies (see Section 1.2 and Section 3.3.1).

Then again, "[b]ecause general relativity is a local theory with no *a priori* restrictions on the global topology, causality violation can be introduced" [234]. Such a global topology could be one where the time coordinate is periodic, as long as it is consistent with Einstein's equations of general relativity. Indeed, solutions with this kind of topology were found (see, *e.g.*, References [131, 165]), and these "causality violations" show up in terms of closed time-like curves (CTCs). Such CTCs allow us to apply the results from the previous chapter to general relativity. Clearly, the existence of CTCs in nature is doubted, however, it cannot be ruled out. This analysis and the next chapter of this thesis show that CTCs are *unproblematic* under various considerations.

5.1 Closed time-like curves in general relativity

Some forms and an historic account of CTCs is presented in the introduction (see Chapter 2.2). Other forms of CTCs within general relativity that are not based on the universes by Lanczos or Gödel, as discussed in the introduction, were discovered, for instance, by Morris, Thorne, and Yurtsever [182] and Gott [138]. The possibly easiest way to understand a setup in general relativity where CTCs arise is the approach by the former (see also [233]). Their approach relies on wormholes: "[I]f the laws of physics permit an advanced civilization to create and maintain a wormhole in space for interstellar travel, then that wormhole can be converted into a time machine with which causality might be violatable" [182]. Wormholes consist of two mouths: By entering one mouth one exits the other (see Figure 5.1). In other words, wormholes connect two space-regions on a different path, potentially shorter than the regular path. Furthermore, one can think of both mouths of a wormhole to have clocks that are initially

[1] "This contradicts my feeling for physics in the most vivid sense."

[2] Translated by Anna Beck [103]: "that we would have to consider as possible a transfer mechanism whereby the achieved effect would precede the cause."

[3] Translated by Anna Beck [103]: "this result, in my opinion, does not contain any contradiction form a purely logical point of view."

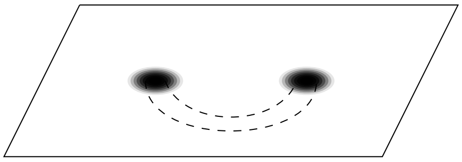


Figure 5.1. The two black ovals denote the two mouths of a wormhole. The distance on the regular space might be longer than the distance through the wormhole (dashed tube).

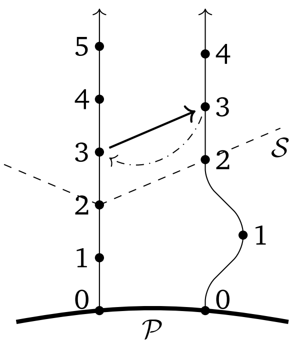


Figure 5.2. This space-time diagram shows the word-lines of the two mouths of a wormhole in the future of some space-like surface $\mathcal{P}$. The left mouth is stationary while the right mouth undergoes the effect of time dilation. The numbers on the word-lines indicate the state of their internal clocks. These points are identified. In the future of the surface $\mathcal{S}$ (the dashed lines describe the future light cone of point 2 on the left), a CTC comes into existence. For instance, if one travels in ordinary space-time from point 3 from the left world-line to the right one, and enters the right mouth, then one exits the wormhole at point 3 on the left world-line again. This CTC is indicated by the arrows.

synchronized, and that if one enters one mouth at some time t, then one exits the other mouth at time t, and *vice versa*; without loss of generality we assume that traveling through the wormhole is instantaneous. By taking one mouth of a wormhole on a rocket and by traveling at some very high velocity, one can apply a time dilation (the twins-paradox). Then, by coming back next to the other mouth, the wormhole describes a CTC (see Figure 5.2). The mouth that traveled is younger when compared to the other mouth. By entering the mouth that did not travel, one time-travels to the future, and by entering the mouth that did travel, one time-travels to the past.

From the logical point of view, the grandfather antinomy could easily be implemented by having access to CTCs as described here. Novikov suggested (see Section 1.2 and Section 2.2) that series of contradictory events (*e.g.*, the grandfather antinomy) simply do not arise. Does this mean that some initial conditions are *ruled* out because they lead to inconsistent dynamics? This would be rather odd: It would mean that the existence of CTCs in our future would prohibit us to perform operations that would otherwise be possible.

5.2 Logical and physical principles

The *Novikov principle of self-consistency*, as briefly discussed above, is the following.

Principle 1 (Novikov's principle of self-consistency). "[E]vents on a CTC are already guaranteed to be self-consistent [...]" [123].

This means that events do not only have to be consistent with their past, but also with their *future* (in a CTC, future events can influence past events). Basically, on a CTC, every event is *before as well as after* every other event on the same CTC. So, the principle of self-consistency asserts that "a local solution to the equations of physics can occur in the real Universe only if it can be extended to be part of a global solution" [123]. By this principle, inconsistent dynamics do not arise *at all*; they are *forbidden*. However, this principle might have severe implications. The problem is that "[i]n general, self-consistency constraints the initial data [...]" [123]. As an illustration, suppose that someone tries to kill his or her grandfather before this person was even born (see Section 1.2 and Section 3.3.1). Now, instead of killing the grandfather directly, this person programs a robot to do so. This robot thereafter travels to the past, finds the grandfather, and kills him. If the robot would succeed, then we have the logical problem of the grandfather antinomy. Now, since such an *inconsistent* series of events is excluded by this self-consistency principle, it means that the person who plans to assassinate his or her grandfather is *restricted* in programing the robot in such a way that it will *not* kill the grandfather. Can this issue be resolved in the sense that we do not have to impose constraints on the initial data? This questions suggests another principle: the *no-new-physics principle*.

Principle 2 (No-new-physics principle). The physical laws, and in particular the set of possible physical operations, in a local space-time region remain the same no matter whether CTCs exist *outside* of that region or not. By the *weak* no-new-physics principle we replace *outside* with *in the future*.

This principle can be understood as a generalization of Einstein's relativity principle which states that the physical laws are invariant for all reference frames. Echeverria, Klinkhammer, and Thorne [97] gave evidence that in the presence of CTCs in the *future* of a local region, the *weak* no-new-physics principle and Novikov's principle of self-consistency can simultaneously be uphold. These authors looked for an initial trajectory of a billiard ball that enters a CTC in its future, such that Novikov's principle of self-consistency would forcefully rule out *that* initial trajectory — and by this, the *weak no-new-physics* principle wold be violated. Surprisingly, they did not find any such contradictory initial trajectory. In contrast to the weak principle, the (strong) principle

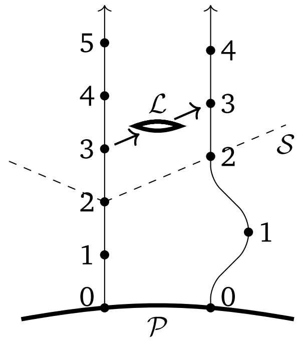


Figure 5.3. A CTC comes into existence in the future of some space-like surface $\mathcal{P}$ on which some initial conditions are set. Furthermore, an experimenter sitting in some local region $\mathcal{L}$ can influence the system that travels on the CTC.

asserts that the set of possible operations remains invariant also in the presence of a CTC *around* such a local region (see Figure 5.3). There, Friedman *et al.* [123] conclude that "initial data [. . .], where the CTCs reside, are constrained."

Besides these principles above, we can formulate another principle that one wishes to uphold.

Principle 3 (Uniqueness principle)**.** There exists a *unique* self-consistent solution.

This principle, in the general relativity setting, basically reflects the Cauchy problem. If all initial data has been provided, then there exists a *unique* solution to the dynamics. Echeverria *et al.* [97] showed that "[i]n contrast with one's naive expectation that dangerous trajectories might have multiplicity zero and thereby make the Cauchy problem ill posed ('no solutions'), it is shown that *all* dangerous initial trajectories in a wide class have *infinite* multiplicity and thereby make the Cauchy problem ill posed in an unexpected way: 'far too many solutions'" (see also Reference [122]). This principle can also be understood as a requirement to avoid the information antinomy (see Section 3.3.1). In the CTC model we are going to present, Principle 3 is show to be a *consequence* of the Principles 1 and 2. So, it is not a principle in the sense of an axiom, but a property that we wish to have — and we get it for free.

Note the close relation of these three principles to the assumptions used in the previous chapter. There, two of the main assumptions where that all local operations are possible (Assumption (C)) and that there exists a global probability distribution over all non-free variables (Assumption (LC)). Assumption (C) can be identified with the principle of no-new-physics, and Assumption (LC) with Novikov's self-consistency principle. Then again, these assumptions lead us to conclude that, in the deterministic case, there exists a *unique* fixed point (Theorem 5) — this can be identified with Principle 3. In the spirit of the previous chapter, we introduce this following definition:

Definition 16 (Logically concistent CTC)**.** A CTC that satisfies the Principles 1, 2, and 3 is called *logically consistent*.

Finally, we formulate a last principle that one wishes to uphold.

Principle 4 (Reversible dynamics)**.** All dynamics can be embedded into reversible dynamics where potentially additional degrees of freedom are considered.

This principle is satisfied by all our current physical theories (even in quantum theory, if one takes the relative state formulation — see Section 2.1.5), and has a natural motivation from thermodynamics: If the principle would be violated, *e.g.*, if some degrees of freedom are miraculously lost (without heat dissipation), then the second law can be violated. This is nicely illustrated by Maxwell's demon [177] and by the resolution of the paradox by Bennett [45]. See also the article by Wolf [250] for a nice account of reversibility and thermodynamics. Just as the uniqueness principle follows from the first two, this one does so as well. Reversible dynamics allow one to tell a *story* and to answers questions like: What is the trajectory of the objects, how do they interact? Furthermore, one can think of an implementation of such a CTC with billiard balls; very much in the spirit of the work by Echeverria *et al.* [97]. In such a case, the operations are implemented by placing walls in space, where the billiard balls get reflected and potentially collide with themselves and with their *younger* selfs.[4]

5.3 Logically consistent closed time-like curves

We present a novel model of closed time-like curves. Similar to Echeverria, Klinkhammer, and Thorne [97], we assume a space-like surface $\mathcal{P}$ where initial conditions are set. In the future of that space-like surface, a CTC comes into existence, *e.g.*, by applying the twins paradox to a wormhole (see above). The findings of Echeverria *et al.* were, as described before, that the existence of a CTC in the future of $\mathcal{P}$ does not change physics at $\mathcal{P}$; the *weak* no-new-physics principle is uphold. Here, we go beyond the studies of Echeverria *et al.* in the sense that we ask whether CTCs are compatible with *arbitrary* local operations in *arbitrary* locations (see Figure 5.3). In principle, one could imagine an experimenter sitting in-between the two mouths of a wormhole, and by this act on the systems that travels on the CTC. Such interactions might allow an experimenter to influence the billiard balls that were studied in the works of Echeverria *et al.* to introduce a contradiction. In order to cover these cases, we additionally assume the existence of a local region $\mathcal{L}$ where experimenters can perform arbitrary operations (see Figure 5.3). Now, by the no-new-physics principle, not only physics at $\mathcal{P}$ but also physics at $\mathcal{L}$ should remain unchanged if there exists a CTC in the future of $\mathcal{P}$ and

[4]This has been explored by Fredkin and Toffoli [120] in the setup *without* CTCs to demonstrate the universality of such a model.

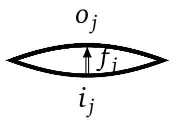


Figure 5.4. Visualization of a local region j in a space-time diagram. The lower arc is the *past* boundary, and the top arc is the *future* boundary of that region. The boundaries are assumed to be space-like. The states on these boundaries are i_j and o_j. Within the local region, *any* deterministic operation from $\mathcal{I}_j$ to $\mathcal{O}_j$ is allowed.

around $\mathcal{L}$. In this study, we do not consider specific physical objects, like billiard balls, but look at classical deterministic dynamics at an abstract level.

The results we show are that there *exist deterministic CTCs that obey the requested principles* (see Principles 1, 2, 3). Furthermore, such CTCs can be made *reversible* (see Principle 4). By the latter, the dynamics can also be read in the opposite direction — just as in general relativity.

5.3.1 The model

We consider a *fixed* number N of distinct local space-time regions (called *local regions*) where every region is composed of a *past* space-like boundary and a *future* space-like boundary. It is *crucial* that the number of local regions is *fixed* — as will become clear later. Every local region is labeled by an integer $1 \leq j \leq N$. The state space of the past space-like boundary of some region j is denoted by $\mathcal{I}_j$ (input), and the state space of the future space-like boundary of the same region is $\mathcal{O}_j$ (output). These spaces, as opposed to the study in the previous chapter, could potentially be continuous, *e.g.*, $\mathcal{O}_j = \mathbb{R}$. States at these boundaries are denoted by $i_j \in \mathcal{I}_j$ and $o_j \in \mathcal{O}_j$. From the no-new-physics principle, we assume that *any* deterministic local operation f_j from $\mathcal{I}_j$ to $\mathcal{O}_j$ is implementable in the region j. Thus, a local region can be visualised as in Figure 5.4. Let $\mathcal{L}_j := \{f_j : \mathcal{I}_j \to \mathcal{O}_j\}$, *i.e.*, $\mathcal{L}_j$ is the set of *all* local operations in that region. Since a local region could contain additional degrees of freedom besides of those on the space-like boundaries, an operation f_j could be a "reduced" local operation where more degrees of freedom are considered. Thus, f_j is not assumed to be reversible. We employ the same notion as in the previous chapter: Collection of objects are represented by simply dropping the index, *e.g.*, $\mathcal{L} := \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_N$. Note that $\mathcal{L}$ is not the set of all functions from $\mathcal{I}$ to $\mathcal{O}$, but only those of the form $f = (f_1, f_2, \ldots, f_N)$. Whatever happens *outside* of these local regions is modeled by another function w. This function describes the dynamics of those degrees of freedom only that are at the boundary of the local regions. Other degrees of freedom are ignored. Furthermore, this dynamics potentially contains a CTC. Thus, w is a function

$$w : \mathcal{O} \to \mathcal{I} .$$

We call this function *process function*.[5]

The no-new-physics principle implies that all operations $f \in \mathcal{L}$ are allowed, and Novikov's principle of self-consistency implies that for every such operation f there exists a consistent evolution. This is formalized as the existence of a fixed point

$$i = (i_1, i_2, \ldots, i_N)$$

for every choice of f:

$$\forall f \in \mathcal{L}, \exists i \in \mathcal{I} : i = w \circ f(i) . \tag{5.1}$$

From this, the uniqueness principle follows:

Theorem 11 (Unique fixed point)**.** *For every choice of local operations $f \in \mathcal{L}$ and for every function w that satisfies Equation* (5.1)*, the fixed point of $w \circ f$ is* unique*. In other words: The uniqueness principle follows from Novikov's principle and from the no-new-physics principle.*

Proof. Let $P[N]$ be the proposition of the theorem where the number of local regions is fixed to N. We prove this theorem by induction over the number of regions N. The base case is $P[1]$, and the induction step is $P[N-1] \implies P[N]$.

In the case where we consider one local region only, the process function w must be a *constant*. Otherwise, there would exist a local operation $f \in \mathcal{L}$ such that $w \circ f$ would not have any fixed point. Assume towards this contradiction that w is not a constant. This means there exist at least two distinct $o, p \in \mathcal{O}$ such that

$$k = w(o) \neq w(p) = \ell .$$

Now, we construct the local operation f

$$\begin{aligned} f : \mathcal{I} &\to \mathcal{O} \\ i &\mapsto f(i) = \begin{cases} o & \text{for } i \neq k \\ p & \text{otherwise.} \end{cases} \end{aligned}$$

By that, the composed function $w \circ f$ has *no* fixed point.

Since w is a constant, for every $f \in \mathcal{L}$, the composed function $w \circ f$ has a *unique* fixed point. This proves the base case. Furthermore, this proves that if the input to a local region is a function of the output of the *same* region, then there exists a local operation such that the composed function has *no* fixed point.

[5]This function can also be derived by starting from some function ω as a function of all local operations only (see Reference [36]).

Towards proving the induction step, we define the *reduced process function* w^{f_N} of $w \equiv (w_1, w_2, \dots, w_N)$ as

$$\begin{aligned} w^{f_N} &\equiv \left(w_1^{f_N}, \dots, w_{N-1}^{f_N}\right) \\ w_j^{f_N} &: \mathcal{O}_1 \times \cdots \times \mathcal{O}_{N-1} \to \mathcal{I}_j \\ & o' \mapsto w_j\left(o_1', \dots, o_{N-1}', f_N\left(w_N\left(o'\right)\right)\right). \end{aligned}$$

This *reduced process function* w^{f_N} is the process function w where the local operation the local region N is *fixed* to f_N. Since by the above considerations (base case), the input to a local region must be *independent* of the output of the same region, the value $f_N(w_N(o'))$ is well-defined.

We now show that the *reduced process function* w^{f_N} satisfies Equation (5.1). Formally, we show

$$\begin{gathered} \forall f \in \mathcal{L}, \exists i \in \mathcal{I} : i = w \circ f(i) \\ \Longrightarrow \\ \forall f' \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_{N-1}, f_N \in \mathcal{L}_N, \exists j \in \mathcal{I}_1 \times \cdots \times \mathcal{I}_{N-1} : j = w^{f_N} \circ f'(j). \end{gathered}$$

For every region $k \in \{1, \dots, N-1\}$, the fixed point of $w \circ f$ restricted to that region is i_k. We show that i_k is also the fixed point of region k of $w^{f_N} \circ (f_1, \dots, f_{N-1})$:

$$w_k^{f_N}\left(f_1(i_1), \dots, f_{N-1}(i_{N-1})\right) = w_k\left(f_1(i_1), \dots, f_{N-1}(i_{N-1}), f_N(i_N)\right) = i_k.$$

Now, we show the contrapositive of the induction step: $\neg P[N] \implies \neg P[N-1]$. For that, assume that $P[N]$ is false. This means there exists a function $f \in \mathcal{L}$, such that $w \circ f$ has more than one fixed point. Let $a \neq b$ be two fixed points of $w \circ f$, and assume without loss of generality that they differ in the first component $a_1 \neq b_1$. From this follows that w^{f_N} has also two fixed points $a' = (a_1, \dots, a_{N-1})$ and $b' = (b_1, \dots, b_{N-1})$. This concludes the proof. □

5.3.2 Reversibility

In the previous chapter, we showed that environments from the deterministic-extrema polytope (set $\mathcal{D}$) can be made reversible. In this chapter on CTCs, we look at deterministic CTCs described by function. Thus, we could simply invoke Theorem 7, and by this, every logically consistent CTC could be made reversible. However, that theorem holds for discrete and finite degrees of freedom. Here, we essentially show the same theorem in the relativity setting and for continuous variables. In the same spirit as Theorem 7, we add two local regions: a source and a sink. The source (see Figure 5.5a) consists of a single space-like boundary. At that boundary, initial conditions are *set*. The sink (see Figure 5.5b) consists of single space-like boundary as well. In contrast to the source, the sink cannot set initial conditions, but can only *observe* the degrees of freedom of

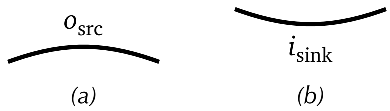


Figure 5.5. (a) A source is a space-like boundary that is an output-only region (emitter). (b) A sink is a space-like boundary that is an input-only region (receiver).

that boundary. These regions can be understood as a special case of the local region (see Figure 5.4): Here, the degrees of freedom of the output or of the input region are nonexistent.

By the following theorem, the reversibility principle is seen to follow from Novikov's principle of self-consistency and from the no-new-physics principle.

Theorem 12 (All logically consistent CTCs can be made reversible)**.** *Every function*

$$w : \mathcal{O} \to \mathcal{I}$$

that satisfies Condition (5.1) *can be embedded into a reversible function*

$$w' : \mathcal{O} \times \mathcal{O}_{src} \to \mathcal{I} \times \mathcal{I}_{sink}$$

that satisfies Condition (5.1) *as well. The CTC w is recovered from w' by some $\tilde{o} \in \mathcal{O}_{src}$ where*

$$\forall o \in \mathcal{O} : w'(o, \tilde{o}_{src}) = (w(o), g_{sink}(o))$$

for some invertible g_{sink}.

Proof. This proof uses Theorem 11. We set $\mathcal{O}_{\mathrm{src}} \equiv \mathcal{I}$ and $\mathcal{I}_{\mathrm{sink}} \equiv \mathcal{O}$. The reversible CTC w' is defined as

$$w' \equiv (w'_1, w'_2)$$

with

$$\begin{aligned} w'_1(o, o_{\mathrm{src}}) &= w(o) + o_{\mathrm{src}} = i\,, \\ w'_2(o, o_{\mathrm{src}}) &= o = i_{\mathrm{sink}}\,. \end{aligned}$$

The inverse $w'^{-1} \equiv (w_1'^{-1}, w_2'^{-1})$ of w' is

$$\begin{aligned} w_1'^{-1}(i, i_{\mathrm{sink}}) &= i_{\mathrm{sink}} = o\,, \\ w_2'^{-1}(i, i_{\mathrm{sink}}) &= i - w(i_{\mathrm{sink}}) = o_{\mathrm{src}}\,. \end{aligned}$$

We need to show that w' satisfies Condition (5.1). The local operation of the source region has the form $f_{\mathrm{src}} : \varnothing \to \mathcal{O}_{\mathrm{src}}$, where $\varnothing$ is the empty set. Such an operation can be understood as a state preparation, and is fully characterized by a state $o_{\text{src}} \in \mathcal{O}_{\text{src}}$. The local operation of the sink region is $f_{\text{sink}} : \mathcal{I}_{\text{sink}} \to \emptyset$ (this can be interpreted as a measurement); thus, we can just omit it. With the two additional regions, Condition (5.1) becomes

$$\forall (f, o_{\text{src}}) \in \mathcal{L} \times \mathcal{O}_{\text{src}}, \exists (i, i_{\text{sink}}) \in \mathcal{I} \times \mathcal{I}_{\text{sink}} : (i, i_{\text{sink}}) = w'(f(i), o_{\text{src}}) .$$

This is the same as the requirement that

$$\begin{aligned} i = w_1'(f(i), o_{\text{src}}) &= w(f(i)) + o_{\text{src}} , \qquad \text{and} \\ i_{\text{sink}} = w_2'(f(i), o_{\text{src}}) &= f(i) . \end{aligned}$$

The latter equation has a fixed point: $i_{\text{sink}} = f(i)$. To see that the former equation has a fixed point, we rewrite the operation $+o_{\text{src}}$ as a function:

$$\begin{aligned} \pi_{o_{\text{src}}} : \mathcal{I} &\to \mathcal{I} , \\ i &\mapsto i + o_{\text{src}} . \end{aligned}$$

Thus, we have

$$i = \pi_{o_{\text{src}}} \circ w \circ f(i) .$$

Since i is supposed to be a fixed point, we can shift the operations and ask for a fixed point i' of

$$i' = w \circ f \circ \pi_{o_{\text{src}}}(i') .$$

Such a fixed point *must* exist because w is assumed to have a fixed point for *every* local operation f; the operation $f \circ \pi_{o_{\text{src}}}$ is a valid local operation as well, *i.e.*, $f \circ \pi_{o_{\text{src}}} \in \mathcal{L}$. □

5.3.3 Example of a logically consistent closed time-like curve

This example is based on one of the extremal points of the polytope studied in Section 4.4 (see also Section 4.6.2). Assume the number of local regions to be $N = 3$. For the sake of better presentation, we rename these three local regions S_1, S_2, S_3 with Alice, Bob, and Charlie. We also use different symbols for the states and the state spaces. The state space on Alice's past boundary is $\mathcal{X}$, and a state is given by $x \in \mathcal{X}$. For her future boundary, we use the state space $\mathcal{A}$ with a state $a \in \mathcal{A}$. Respectively, the state spaces of Bob's past and future boundary are $\mathcal{Y}, \mathcal{B}$, and for Charlie: $\mathcal{Z}, \mathcal{C}$. Here, we look at continuous variables, *i.e.*, we assume all state spaces to be $\mathbb{R}$. Thus, a CTC in this setting, is a function

$$\begin{aligned} w : \mathbb{R}^3 &\to \mathbb{R}^3 , \\ (a, b, c) &\mapsto (x, y, z) . \end{aligned}$$

In particular, and in the spirit of the extremal point presented in Section 4.6.2, the CTC w implements the dynamics given by

$$x = \Theta(-b)\Theta(c)\,,$$
$$y = \Theta(-c)\Theta(a)\,,$$
$$z = \Theta(-a)\Theta(b)\,,$$

where $\Theta(t) = 1$ for some $t > 0$, and 0 otherwise. So, the *sign* of every output determines the signaling direction between the remaining two regions. For instance, if a is positive, then $y = \Theta(-c)$, yet $z = 0$ — Charlie's input *does not* depend on Bob's output. Then again, for a negative, we have $y = 0$ and $z = \Theta(b)$ — now, Charlie's input depends on Bob's output but Bob's input is a constant.

Now, we embed w into a *reversible* CTC. For that purpose, we have to introduce two local regions: a source which has a future boundary only, and a sink which has a past boundary only. The state space on each boundary is $\mathbb{R}^3$. The reversible CTC w' is

$$w' : \mathbb{R}^6 \to \mathbb{R}^6\,,$$
$$(a, b, c, e_0, e_1, e_2) \mapsto (x, y, z, s_0, s_1, s_2)\,,$$

where (x, y, z) are defined as

$$x = \Theta(-b)\Theta(c) + e_0\,,$$
$$y = \Theta(-c)\Theta(a) + e_1\,,$$
$$z = \Theta(-a)\Theta(b) + e_2\,,$$

and where the sink obtains the output of the parties:

$$s_0 = a\,, \quad s_1 = b\,, \quad s_2 = c\,.$$

This setup is shown in Figure 5.6.

Alice, Bob, and Charlie are allowed to implement *any* dynamics from a real degree of freedom to another real degree of freedom. The state at the *past* boundary of everyone *depends non-trivially on the state at the future boundary of the remaining two parties*. This leads to a *causal loop*: By considering their local dynamics, which links x to a, y to b, and z to c, we see that the states at the past (future) boundaries depend on *themselves*. That is, (x, y, z) depends on (a, b, c), which in turn determines (x, y, z). This loop leads to conclude that every party is *both, in the past and in the future of every other party*. Yet, all four principles required are fulfilled. Moreover, the last two principles are seen to follow from the first two. Thus, we have presented a CTC that is completely *harmless* from the logical and physical point of view.

We briefly put an eye on the assumption that we *fixed* the number of parties beforehand. Imagine, local regions are allowed to pop up everywhere, and that in every local region, the no-new-physics principle holds. By this, we can just take the CTC described

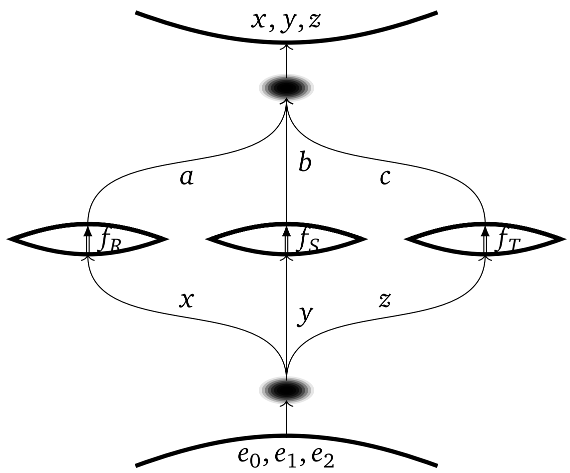


Figure 5.6. A reversible CTC. Alice (R) can invoke an arbitrary operation from $\mathbb{R}$ to $\mathbb{R}$ which describes the dynamics f_R of real degrees of freedom from her past to her future boundary. The same holds for Bob (S) and Charlie (T). The ovals denote the CTC, where all degrees of freedom are scrambled up.

in Figure 5.6 and place an additional local region just before the CTC. If an experimenter would act freely at such a local region, then she could just undo the “scrambling up” of the degrees of freedom, in such a way that a contradiction arises. However, if we assume a *fixed* number of local regions, then *every* local region is considered, and the presented results can be shown.

Such a CTC, in the finite-dimensional case, is also consistent with quantum mechanics, as is shown in Section 4.8.2. This is done by replacing every classical space with a Hilbert space. In the quantum setting, a CTC is reversible if it describes a *unitary* process from the output spaces of the parties to their input spaces. A CTC, then again, is logically consistent, if for all *all* quantum operations within the local regions, no contradiction arises.

5.4 Other models of closed time-like curves

We present two models of closed time-like curves that do not incorporate the formalism from general relativity, but where such CTCs are simply assumed to exist. These two models are different in the sense that they solve the Principles 1, 2, 3 in different ways, and lead to different evolutions. The reversibility principle is violated in both models. In Section 5.5, we compare these models to our model of CTCs.

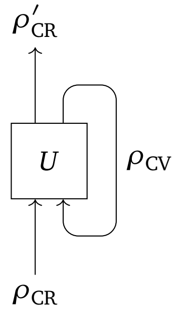


Figure 5.7. Closed time-like curve in the Deutsch model: The state ρ_{ctc} of the causality-violating system is a fixed point of the evolution.

5.4.1 Deutschian closed time-like curves

Deutsch [93] analyzed CTCs without taking the approach of general relativity, but by abstracting away the relativity formalism and by considering some degrees of freedom only. The main message of Deutsch is that CTCs can be modeled consistently within quantum theory. The standpoint of quantum theory is taken, because Deutsch shows in the same work that if one takes classical systems in terms of bits, then inconsistencies *do arise*.

The basic idea of his model is the following. Assume two systems, CR and CV, undergo a joint evolution U. The system CR, is *causality respecting*, which means that it does *not* travel on a CTC. The other system, CV, is *causality violating*, which means that is *does* travel on a CTC. The states ρ_{CR}, ρ_{CV} of these systems describe the values of the degrees of freedom. After the evolution, the states of these systems are ρ'_{CR} and ρ'_{CV}, and the system CV *travels to the past* where it enters the joint evolution (see Figure 5.7). At this point, Deutsch introduces the *condition of consistency*, namely that the causality-violating system is in the *same state before and after* the evolution, *i.e.*, $\rho_{CV} = \rho'_{CV}$ must hold. This condition ensures that the system CV is not in contradictory states. Thus, by plugging in some ρ_{CV} where system CV is in state ρ_{CV} *after* the evolution U, one can describe the evolution of system CR from ρ_{CR} to ρ'_{CR} (see Figure 5.7). The initial conditions of that model are the state of the system CR and the choice of the evolution U.

Let us briefly look at the case where the systems are modeled by *bits*, *e.g.*, where

$$\rho_{CR}, \rho'_{CR}, \rho_{CV} \in \{0,1\},$$

and thus where the evolution U takes the form

$$U : \{0,1\} \times \{0,1\} \to \{0,1\} \times \{0,1\}.$$

We give an example of an *unproblematic* evolution, of an evolution where the possible input states for system CR are *restricted*, and one where *every input* on CR leads to a contradiction. These examples show that *some initial conditions have to be excluded* in the classical case — which contradicts the principle that physics locally, should not depend on the global embedding (see Principle 2). For the first example, assume the following evolution

$$U : (\rho_{\mathrm{CR}}, \rho_{\mathrm{CV}}) \mapsto (\rho_{\mathrm{CR}}, 0) .$$

The state after the evolution is $(\rho_{\mathrm{CR}}, 0)$. Thus, a consistent state for the CV system exists:

$$\rho_{\mathrm{CV}} = 0 .$$

Furthermore, for any ρ_{CR}, the evolution U combined with the CTC describes an evolution from ρ_{CR} to itself.

The second example is

$$U : (\rho_{\mathrm{CR}}, \rho_{\mathrm{CV}}) \mapsto (\rho_{\mathrm{CR}}, \rho_{\mathrm{CR}} \oplus \rho_{\mathrm{CV}}) .$$

For $\rho_{\mathrm{CR}} = 0$ every ρ_{CV} is *consistent*: $U(0, \rho_{\mathrm{CV}}) = (0, \rho_{\mathrm{CV}})$. However, for $\rho_{\mathrm{CR}} = 1$, every choice of state on the second system is contradictory: $U(1, \rho_{\mathrm{CV}}) = (1, \rho_{\mathrm{CV}} \oplus 1)$. Here, an experimenter is restricted in setting up the initial conditions in such a way that $\rho_{\mathrm{CR}} = 0$; this is the only allowed initial state of the causality-respecting system.

The last example is the unconditional bit-flip operation on the CV system:

$$U : (\rho_{\mathrm{CR}}, \rho_{\mathrm{CV}}) \mapsto (\rho_{\mathrm{CR}}, \rho_{\mathrm{CV}} \oplus 1) .$$

Here, no choice of the initial state of the causality-respecting system exists such that the evolution is consistent.

In contrast to this classical case just discussed, Deutsch shows that in the quantum case, there *always* exists a consistent evolution. In this setup, the systems CR and CV are quantum systems and U is a map from quantum states to quantum states. Furthermore, we assume U to be unitary. Let the states of the systems CR, CV, be the density operators $\rho_{\mathrm{CR}} \in \mathcal{S}(\mathcal{H}_{\mathrm{CR}})$ and $\rho_{\mathrm{CV}} \in \mathcal{S}(\mathcal{H}_{\mathrm{CV}})$. The consistency condition in the quantum setup is thus

$$\rho_{\mathrm{CV}} = \mathrm{Tr}_{\mathrm{CR}}\left(U\left(\rho_{\mathrm{CR}} \otimes \rho_{\mathrm{CV}}\right)U^{\dagger}\right) , \tag{5.2}$$

and the final state ρ'_{CR} of the causality-respecting system is computed by

$$\rho'_{\mathrm{CR}} = \mathrm{Tr}_{\mathrm{CV}}\left(U\left(\rho_{\mathrm{CR}} \otimes \rho_{\mathrm{CV}}\right)U^{\dagger}\right) . \tag{5.3}$$

The consistency condition (5.2) ensures that the grandfather antinomy does not arise. But what about the information antinomy? It is easy to see that for some choices of U and ρ_{CR} there exists *multiple* states ρ_{CV} that satisfy the consistency condition (5.2). To overcome this latter antinomy, Deutsch suggests to select the ρ_{CV} state as the one that is consistent in the above sense and *maximizes* the entropy. Intuitively, the system traveling on the CTC should contain as little information as possible. However, Deutsch [93] himself admits that by doing so "the second [antinomy] is mitigated." This model is henceforth called D-CTC.

D-CTC satisfies Novikov's principle of self-consistency, and the no-new-physics principle. The uniqueness principle is satisfied in an *ad-hoc* manner — by selecting the fixed point with maximal entropy. The last principle, the reversibility principle, is not satisfied. Because of the trick to single out a single state on the CTC (to fulfill the uniqueness principle), dynamics are non-linear, as is shown in the next paragraph. By that, if one sends an *unknown* quantum state through the D-CTC, then that state might undergo a transformation which is *not reversible*.

Let us point at some properties of the D-CTC model. First of all, it is clear that the evolution of the causality-respecting system is *non-linear*: The output ρ'_{CR} depends on the input ρ_{CR} and on the fixed point ρ_{CV}, which depends on the input ρ_{CR} as well. This non-linearity is scrambled up with the selection of the CTC state with the highest entropy. It has been shown that this model allows to distinguish non-orthogonal quantum states perfectly [60, 61], to clone perfectly [11, 61], to violate the Holevo bound [60], and that a computer that is modeled with D-CTCs can solve PSPACE-complete problems in polynomial time [6]. Most strikingly, D-CTCs allow to solve the Halting problem [7]; D-CTCs are *super*-Turing machines.

Aaronson and Watrous [6] furthermore showed that in the *classical* case, such a computer could solve PSPACE-complete problems in polynomial time as well. Here, by classical we mean that the systems involved are not quantum states but *probability distributions*. Note the difference to what we meant by classical at the beginning of this discussion. Deutsch excluded probability distributions, in lieu thereof, he assumed that the state of the classical system is "always an element of a certain fixed basis, the computational basis" [93]. In Aaronson's and Watrous' work, in contrast, a probability distribution is allowed to be a fundamental state, where the probabilities do not reflect mere ignorance. Such states are called "proper mixtures."

5.4.2 Postselected closed time-like curves

Another idea to model CTCs in the field of quantum information is by teleportation [46, 199, 208, 229, 230]. In quantum theory, it is possible to *teleport* a quantum state from one location to another [47]. What we mean by teleport is that the state of the system is *never in-between the two locations*. Rather, classical information has to be sent from one location to the other, and furthermore, both locations need to share an entangled quantum state. In this model of CTCs (P-CTC), a quantum state is teleported to the *past*. Now, this sounds like we would have to send classical information from the future to the past to establish such a CTC. This issue is overcome by the means of *post-selection* — which is essentially the same as sending information to the past.

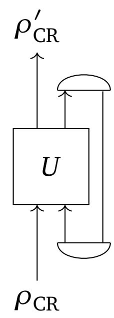


Figure 5.8. Post-selected closed time-like curve: One part of a maximally entangled state undergoes a joint evolution with the causality-respecting system. After that evolution, the causality-violating system is teleported to the past via post-selection.

As in the D-CTC model, we assume two quantum systems, a causality-respecting one (CR) and a causality-violating one (CV). Both systems jointly undergo an evolution, and thereafter, the causality-violating system is teleported to the past, where it enters the evolution (see Figure 5.8). Mathematically, the evolution of the causality-respecting system is expressed by

$$\rho'_{\mathrm{CR}} = \frac{C\rho_{\mathrm{CR}}C^\dagger}{\mathrm{Tr}\left(C\rho_{\mathrm{CR}}C^\dagger\right)}, \qquad \text{with} \qquad C = \mathrm{Tr}_{\mathrm{CV}}\, U\,. \tag{5.4}$$

A detailed description of the P-CTC model is given, *e.g.*, by Brun and Wilde [59].

The P-CTC model of closed time-like curves satisfies, just as D-CTC does, Novikov's principle of self-consistency, and the no-new-physics principle. The uniqueness principle is also satisfied. Yet, just as D-CTC, P-CTC leads to *non-linear* dynamics, posing a problem for the reversibility principle.

The reason for the non-linearity is that the post-selection introduces a renormalization which in general depends on the input state and on the evolution. This model has many similar, yet weaker, features than D-CTC [59]. For instance, it allows to solve problems that are PP-complete in polynomial time [170] (the class PP is assumed to be strictly contained within PSPACE). A classical analogue of the P-CTC model has also been formalized [48]. Its complexity theoretic power is identified by the class $\mathsf{BPP}_{\mathrm{path}}$ [2, 170, 171]. So, this model is inequivalent to D-CTC [171].

5.5 Relations to other CTC models

Our model of logically consistent closed time-like curves is different from the other models of CTCs just presented. We show in which way they are different, and in which way they are equal to our model. This is done in the quantum formalism solely.

One way to relate these models is to take a logically consistent process matrix and to transform it into a P-CTC or D-CTC dynamics. Since process matrices lead to interesting

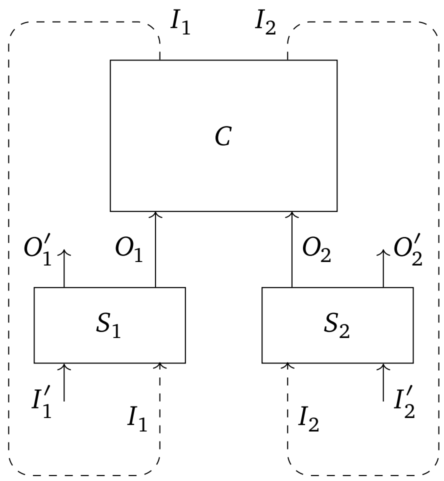


Figure 5.9. The two parties S_1 and S_2 interact with two separate systems. A system of each party undergoes some joint evolution C, and is thereafter sent to the past (dashed lines), where it enters the laboratories of the parties.

dynamics (non-causal correlations, CTCs) for at least two parties, we restrict ourselves to this two-party case.

As it turns out, the process-matrix formalism is a special of the P-CTC model. To see this, we take any logically consistent process matrix and show that it can be cast into the P-CTC model. Then again, we simulate P-CTC dynamics with process matrices and show that some dynamics lead to process matrices that are *not logically consistent*.

For the other CTC model, D-CTC, we show that the dynamics predicted by process matrices are *different* from the D-CTC model. This is done, just as for the comparison with P-CTC, by taking a logically consistent process matrix and by showing that it cannot be cast into D-CTC directly. Some D-CTC dynamics, however, can be simulated in the process matrix framework. We show this by simulating a special case of D-CTC in P-CTC.

There are essentially two ways in how to embed the two parties involved in a process-matrix dynamics in a CTC. Either, both parties are *space-like* separated, or *one party is before the other*. In both cases, we assume that every party interacts with two systems, and that only one system of every party undergoes some CTC dynamics. By that, we obtain a map from the inputs to the outputs (the untouched systems) where the CTC dynamics can be understood as auxiliary. In the first case, after both parties interacted with their respective systems, the systems undergo the CTC dynamics, and enter the laboratories of both parties. This setup is shown in Figure 5.9. In the latter case, one party interacts with a system, which thereafter is sent to the other party. There, the second party interacts with the system, after which, the system falls into the CTC. The CTC produces the input system to the first party (see Figure 5.10).

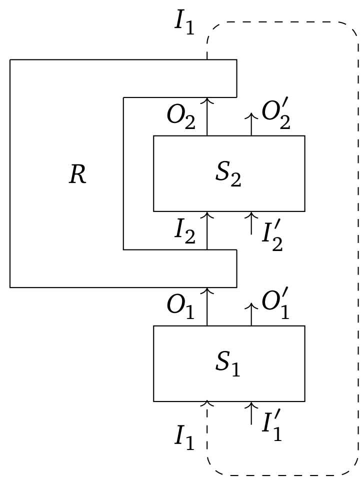


Figure 5.10. Party S_1 first interacts with a system which enters the laboratory of party S_2 at a later time. Around party S_2 some transformation happens (R). After that transformation, the system travels to the past (dashed line), where it enters the laboratory of party S_1.

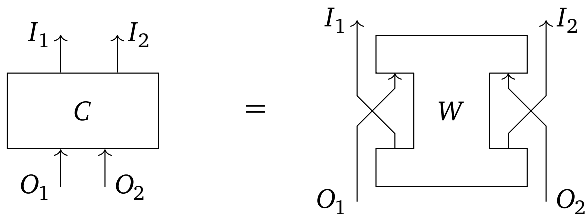


Figure 5.11. By swap operations on both sides one obtains the canonical channel C from a process matrix W.

5.5.1 Canonical transformations

We give a definition of a *canonical channel* and a *canonical comb* which are built from a process matrix W involving two parties. These canonical transformation are then plugged into the P-CTC or D-CTC model. In these definitions, two parties S_1 and S_2 are involved with the respective input systems I_j, I_j' and output systems O_j, O_j', with $j \in \{1,2\}$. The non-prime systems are those that interact with the process matrix. For the canonical transformations, the input and output systems have the matching dimensions $d_{I_j'} = d_{O_j}$ and $d_{I_j} = d_{O_j'}$.

Definition 17 (Canonical channel)**.** The *canonical channel* is obtained by taking a two-party process-matrix $W \in \mathcal{L}(\mathcal{H}_{I_1} \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{I_2} \otimes \mathcal{H}_{O_2})$ and by applying a swap operation for each party (see Figure 5.11). The Choi map of the canonical channel is

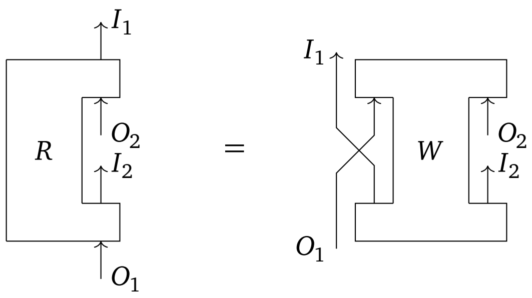


Figure 5.12. By a swap operation on one side one obtains the canonical comb R from a process matrix W.

$$C = W^T$$

where the transposition changes the back-in-time W to a forward-in-time C.

For some input state ρ on the systems O_1, O_2, the canonical channel outputs the state

$$\rho'_{I_1,I_2} = Tr_{O_1,O_2}\left(\left(\rho^T_{O_1,O_2} \otimes \mathbb{1}_{I_1,I_2}\right) C\right) .$$

Since a process matrix is a CPTP map, it is clear that this canonical channel is a CPTP map as well.

Definition 18 (Canonical comb)**.** The *canonical comb* is obtained by taking a two-party process-matrix $W \in \mathcal{L}(\mathcal{H}_{I_1} \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{I_2} \otimes \mathcal{H}_{O_2})$ and by applying a swap operation for one party only (see Figure 5.12). The resulting quantum comb is

$$R = W^{T_{I_1,O_1}} ,$$

where the superscript T_{I_1,O_1} denotes a transposition of the systems I_1 and O_1. This transposition changes the back-in-time flavor on the systems I_1, O_1 to forward-in-time.

Again, since W is a CPTP map, it is clear that R is a comb. Quantum *combs* were introduced in the works by the Pavia group [68–70]. A comb can be seen as a super-operator: It transforms quantum channels to quantum channels, whereas a operator transforms quantum states to quantum states.

5.5.2 From process matrices to P-CTCs

Theorem 13 (Process matrices describe P-CTC transformations)**.** *The dynamics of a process matrix coïncides with the dynamics of the P-CTC model with the canonical channel. The same holds for the canonical comb.*

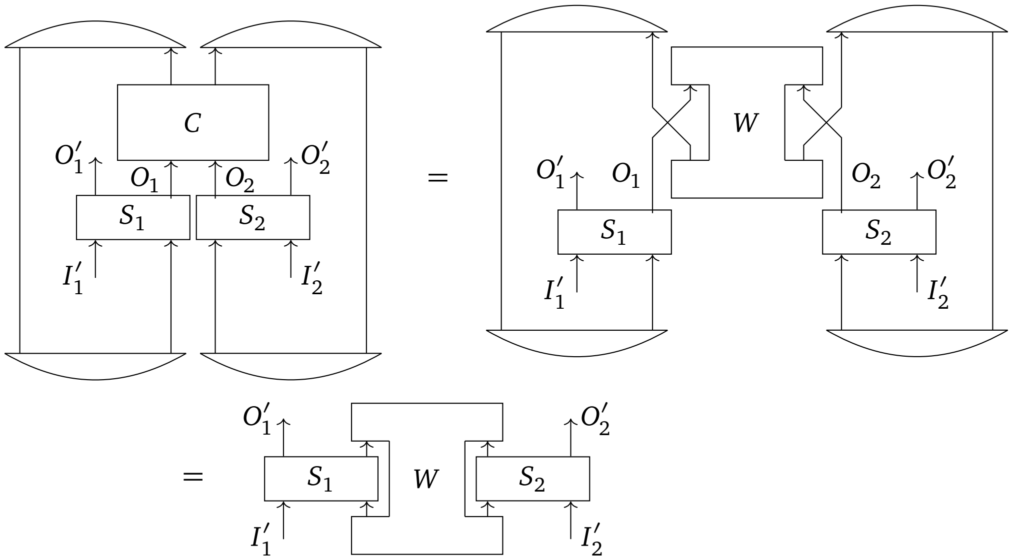


Figure 5.13. The canonical channel in the P-CTC model is the same as the process-matrix W.

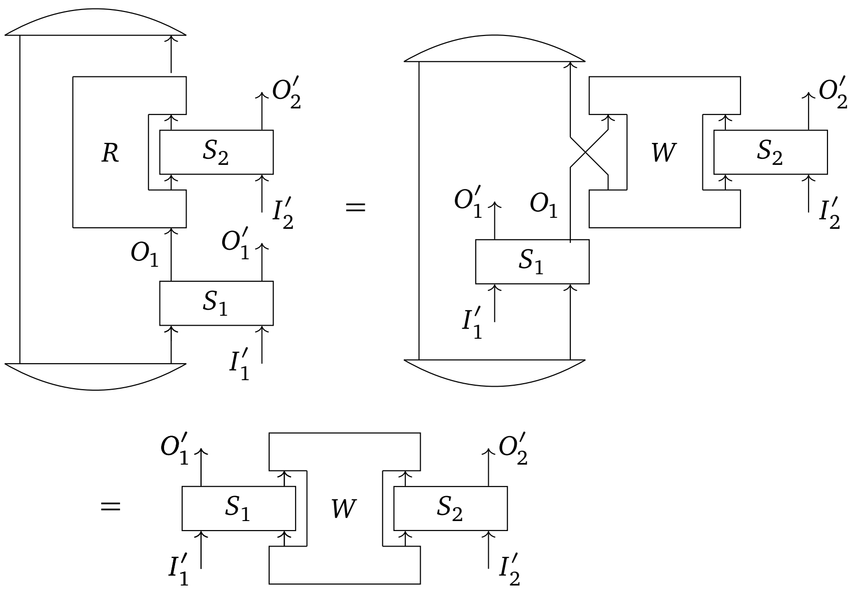


Figure 5.14. The canonical comb in the P-CTC model is the same as the process-matrix W.

Proof. The proof is given by Figures 5.13 and 5.14. The partial ovals on the bottom of the figures represent maximally entangled states, and the partial ovals on the top of the figures the post-selection to the same states. Lines that enter (leave) an entangled state (a post-selection) can be connected. □

Thus, the canonical transformations of every process matrix represent valid P-CTC dynamics.

5.5.3 From process matrices to D-CTCs

In the D-CTC model, the system on the CTC is associated with a density operator — it is always in the product state with the rest. This is the reason why the process-matrix framework yields different predictions for the transformations. The predictions are different in both cases, if we embed the canonical channel, as well as if we embed the canonical comb.

Theorem 14 (Process matrices do not descrie D-CTC transformations). *If process matrices are transformed to canonical channels or canonical combs, then these transformations do not coïncide with the D-CTC predictions.*

Before we prove this theorem, we need to describe the D-CTC dynamics in this setup. For simplicity, we assume that the causality-respecting input systems are trivial, *i.e.*, $d_{I'_1} = d_{I'_2} = 1$. These input systems can simply be embedded into the local operations of both parties.

From process matrices to D-CTCs: Canonical channel

We start by describing the Choi map T_{channel} of the canonical channel embedded in the D-CTC model. The map T_{channel} is a CPTP map from $\tilde{I}$ (the inputs to the parties) to (O', I) (the output of the parties and the output of the canonical channel, see Figure 5.15). According to our notation, we refer to the collection of objects whenever we drop the indices referring to local regions (parties). The spaces of the systems $\tilde{I}_j$ and I_j, for $j \in \{1,2\}$ are isomorphic. Here, as opposed to the discussion of the D-CTC model above (see Section 5.4.1), we use different labels for the systems *entering* and *leaving* the CTC to avoid ambiguities. The reason for this subtlety is that T_{channel} is a Choi map as opposed to a unitary — as above — where it is not ambiguous.

The Choi map T_{channel} is an element from the set $\mathcal{L}(\mathcal{H}_{\tilde{I}} \otimes \mathcal{H}_I \otimes \mathcal{H}_{O'})$. It is constructed by composing the maps of both parties in parallel, and by then composing those in sequence with C. The composition of Choi maps is introduced in the preliminaries (see Appendix). The local operation of party S_j is the Choi map $\tilde{L}_j$ which is an element

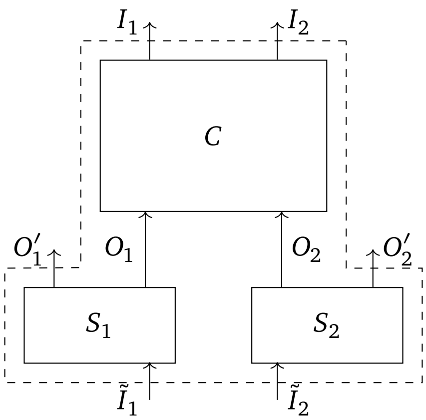


Figure 5.15. The canonical channel C plugged into the D-CTC model yields a new CPTP map T_{channel} (dashed box).

in $\mathcal{L}(\mathcal{H}_{\tilde{I}_j} \otimes \mathcal{H}_{O'_j} \otimes \mathcal{H}_{O_j})$ for $j \in \{0,1\}$. By this, we arrive at

$$\begin{aligned} T_{\text{channel}} &= (\tilde{L}_1 \otimes \tilde{L}_2) \star C \\ &= Tr_O \left(\left(\tilde{L}_1 \otimes \tilde{L}_2 \right)^{T_O} (C \otimes \mathbb{1}_{O'}) \right) \\ &= Tr_O \left(\left(\tilde{L}_1 \otimes \tilde{L}_2 \right)^{T_O} \left(W^T \otimes \mathbb{1}_{O'} \right) \right) . \end{aligned}$$

The D-CTC model then predicts the output state

$$\rho_{O'} = \text{Tr}_{\tilde{I},I} \left(\left(\hat{\rho}_{\tilde{I}}^T \otimes \mathbb{1}_{I,O'} \right) T_{\text{channel}} \right) ,$$

where $\hat{\rho}_{\tilde{I}}$ is the solution to the consistency equation (fixed point):

$$\begin{aligned} \hat{\rho}_I &= \text{Tr}_{\tilde{I},O'} \left(\left(\tilde{\rho}_{\tilde{I}}^T \otimes \mathbb{1}_{I,O'} \right) T_{\text{channel}} \right) , \\ \hat{\rho}_I &= \hat{\rho}_{\tilde{I}} . \end{aligned}$$

From process matrices to D-CTCs: Canonical comb

Here, we construct the Choi map T_{comb} of the canonical comb embedded in the D-CTC model. The local operation of party S_1 is the Choi map $\tilde{L}_1$ which is an element in $\mathcal{L}(\mathcal{H}_{\tilde{I}_1} \otimes \mathcal{H}_{O'_1} \otimes \mathcal{H}_{O_1})$. The Choi map of the local operation L_2 of party S_2, then again, is an element in $\mathcal{L}(\mathcal{H}_{I_2} \otimes \mathcal{H}_{O'_2} \otimes \mathcal{H}_{O_2})$. This Choi map T_{comb} is an element in

$$\mathcal{L}(\mathcal{H}_{\tilde{I}_1} \otimes \mathcal{H}_I \otimes \mathcal{H}_{O'}),$$

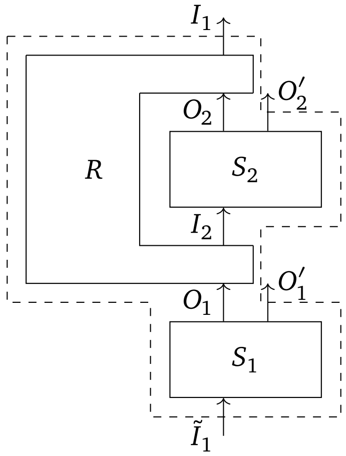


Figure 5.16. The canonical comb R plugged into the D-CTC model yields a new CPTP map T_{comb} (dashed box).

it maps quantum states from $\tilde{I}_1$ to O', I_1 (see Figure 5.16), and is given by

$$\begin{aligned}
T_{\text{comb}} &= \tilde{L}_1 \star R \star L_2 \\
&= \mathrm{Tr}_{I_2,O}\left(\left(\tilde{L}_1^{T_{O_1}} \otimes \mathbb{1}_{I,O_2,O_2'}\right)\left(R^{T_{I_2}} \otimes \mathbb{1}_{\tilde{I}_1,O'}\right)\left(L_2^{T_{O_2}} \otimes \mathbb{1}_{\tilde{I}_1,I_1,O_1,O_1'}\right)\right) \\
&= \mathrm{Tr}_{I_2,O}\left(\left(\tilde{L}_1^{T_{O_1}} \otimes L_2^{T_{O_2}} \otimes \mathbb{1}_{I_1}\right)\left(R^{T_{I_2}} \otimes \mathbb{1}_{\tilde{I}_1,O'}\right)\right) \\
&= \mathrm{Tr}_{I_2,O}\left(\left(\tilde{L}_1^{T_{O_1}} \otimes L_2^{T_{O_2}} \otimes \mathbb{1}_{I_1}\right)\left(W^{T_{I,O_1}} \otimes \mathbb{1}_{\tilde{I}_1,O'}\right)\right) .
\end{aligned}$$

The consistency condition asks for the fixed point

$$\hat{\rho}_{I_1} = \mathrm{Tr}_{\tilde{I}_1,O'}\left(\left(\hat{\rho}_{\tilde{I}_1} \otimes \mathbb{1}_{I_1,O'}\right) T_{\text{comb}}\right) ,$$

with $\hat{\rho}_{I_1} = \hat{\rho}_{\tilde{I}_1}$. The D-CTC model predicts the following state as the output:

$$\rho_{O'} = \mathrm{Tr}_{\tilde{I}_1,I_1}\left(\left(\hat{\rho}_{\tilde{I}_1} \otimes \mathbb{1}_{I_1,O'}\right) T_{\text{comb}}\right) .$$

From process matrices to D-CTCs: Conclusion

In contrast to both cases above (canonical channel and comb), the process-matrix framework predicts the output

$$\rho_{O'} = \mathrm{Tr}_{O,I}\left((L_1 \otimes L_2)(W \otimes \mathbb{1}_{O'})\right) ,$$

where local operation L_j is $\tilde{L}_j$ where the system $\tilde{I}_j$ is renamed I_j. This output *differs* from the predictions of D-CTC in both setups (canonical channel and comb). One example of a process matrix W that yields different predictions for D-CTC is the process matrix described in the work by Oreshkov, Costa, and Brukner [194].

5.5.4 Simulating P-CTCs with process matrices

Here, we show that every P-CTC can be simulated by a process matrix. However, in some cases, the resulting process matrices are *not* logically consistent. This is easily understood: The P-CTC model describe CTCs in the *single* party scenario as well, but the process-matrix framework does not. In this section, we study this *single* party case. Party S has two input systems I, I' and two output systems O, O', where the non-prime systems are *causality violating*, and the prime systems are *causality respecting* (as above). The local operation of party S is the CPTP map expressed as a Choi operator L, which is an element of $\mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_{I'} \otimes \mathcal{H}_O \otimes \mathcal{H}_{O'})$.

Theorem 15 (Simulating P-CTC with process matrices). *The logically-*in*consistent process matrix*

$$W = \rho_{I'}^T \otimes \mathbb{1}_{O'} \otimes |\psi\rangle\langle\psi|_{I,O}\,, \qquad \textit{with}$$
$$|\psi\rangle = \frac{1}{\sqrt{d_I}} \sum_{i=0}^{d_I-1} |i\rangle_I \otimes |i\rangle_O$$

simulates a P-CTC with the dynamics from the system I' to O' The state of output system O' is calculated according to process-matrix formalism with an additional normalization

$$\rho'_{O'} = \frac{\mathrm{Tr}_{I',I,O}(LW)}{\mathrm{Tr}(LW)}\,.$$

Proof. We look at the case where the local operation L is the Choi map of some unitary U. This Choi map is written as

$$L = \left(\mathbb{1}_{I',I} \otimes U_{O',O}\right)|\phi\rangle\langle\phi|\left(\mathbb{1}_{I',I} \otimes U^\dagger_{O',O}\right),$$

where

$$|\phi\rangle = \sum_{k,\ell} |k,\ell\rangle_{I',I} \otimes |k,\ell\rangle_{O',O}\,,$$

with appropriate dimensions. By the following calculation, we see that the expression in the theorem is equal to one used in the P-CTC model (see Equation (5.4)):

$$\begin{aligned}\mathrm{Tr}_{I',I,O}(LW) &\propto \sum_{\substack{i,j,k\\ \ell,m,n}} \mathrm{Tr}\left(|i\rangle\langle k|\rho_{I'}^T\right)\mathrm{Tr}(|j\rangle\langle\ell|m\rangle\langle n|)\,\mathrm{Tr}_O\left(U|j,i\rangle\langle\ell,k|U^\dagger(|m\rangle\langle n|\otimes\mathbb{1})\right)\\ &= \sum_{i,j,k,\ell} \mathrm{Tr}\left(|i\rangle\langle k|\rho_{I'}^T\right)\mathrm{Tr}_O\left(U|j,i\rangle\langle\ell,k|U^\dagger(|\ell\rangle\langle j|\otimes\mathbb{1})\right).\end{aligned}$$

We express $\rho_{I'}$ as

$$\rho_{I'} = \sum_{x,y} \alpha_{x,y}|x\rangle\langle y|\,.$$

Now, we can write

$$\begin{aligned}
\mathrm{Tr}_{I',I,O}(LW) &\propto \sum_{\substack{i,j,k\\ \ell,x,y}} \alpha_{x,y}\,\mathrm{Tr}\left(|i\rangle\langle k|y\rangle\langle x|\right)\mathrm{Tr}_O\left(U|j,i\rangle\langle \ell,k|U^\dagger(|\ell\rangle\langle j|\otimes \mathbb{1})\right)\\
&= \sum_{j,\ell,x,y} \alpha_{x,y}\,\mathrm{Tr}_O\left(U|j,x\rangle\langle \ell,y|U^\dagger(|\ell\rangle\langle j|\otimes \mathbb{1})\right)\\
&= \sum_{\substack{j,\ell,x,y\\ a}} \alpha_{x,y}((\langle a|\otimes \mathbb{1})U|j,x\rangle\langle \ell,y|U^\dagger(|\ell\rangle\langle j|a\rangle\otimes \mathbb{1})\\
&= \sum_{\ell,x,y,a} \alpha_{x,y}((\langle a|\otimes \mathbb{1})U|a,x\rangle\langle \ell,y|U^\dagger(|\ell\rangle\otimes \mathbb{1})\\
&= C\rho_{I'}C^\dagger\,,
\end{aligned}$$

where

$$C = \mathrm{Tr}_O\, U\,.$$

□

5.5.5 Simulating D-CTCs with P-CTCs

Under some restrictions of the local operation, the D-CTC dynamics can be simulated with the P-CTC model. To achieve that, however, we need to introduce some non-linear entanglement-breaking channel into the P-CTC model. Here, we assume a single party with input systems I' and I, and output systems O' and O. The local operation of the party is denoted by U, and is assumed to be a unitary transformation from I, I' to O, O'. As above, the non-prime systems are *causality violating*, while the others are not. The unitary transformations U that we consider are restricted to satisfy the following equation:

$$\begin{aligned}
\rho'_{O'} &= \mathrm{Tr}_O\left(U\left(\rho_{I'}\otimes\frac{\mathbb{1}_I}{d_I}\right)U^\dagger\right)\\
&= \mathrm{Tr}_O\left(U\left(\rho_{I'}\otimes\rho'_I\right)U^\dagger\right),
\end{aligned}$$

with $\rho'_{O'} = \rho'_I$. In words, the fixed point ρ' is obtained by a single use of the unitary on the maximally-mixed state. This boils down that the CPTP map from I to O, *i.e.*, the CPTP map on the causality-violating systems, if written by Kraus operators $\{E_k\}_{\mathcal{K}}$, satisfies

$$\sum_{k\in\mathcal{K}} E_k E_k^\dagger = \sum_{k,k'\in\mathcal{K}} E_k E_{k'} E_{k'}^\dagger E_k^\dagger\,. \tag{5.5}$$

This equations, expressed in words, reads that the outputs of the CPTP map after the first and a second application on the maximally mixed state are equal.

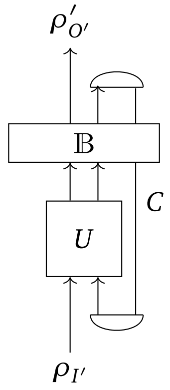


Figure 5.17. Simulating D-CTC with P-CTC.

Theorem 16 (Simulating D-CTCs with P-CTCs). *In the post-selected model of CTCs, an additional system C is required with which the output on the CTC is teleported to the past. The evolution of a system interacting with a D-CTC, with the constraint* (5.5)*, is simulated in the P-CTC model if after the local operation the entanglement-breaking channel* $\mathbb{B}$

$$\begin{aligned}\mathbb{B} : \mathcal{S}(\mathcal{H}_{O'} \otimes \mathcal{H}_O \otimes \mathcal{H}_C) &\to \mathcal{S}(\mathcal{H}_{O'} \otimes \mathcal{H}_O \otimes \mathcal{H}_C)\\ \sigma &\mapsto \operatorname{Tr}_{O',C} \sigma \otimes \operatorname{Tr}_O \sigma\end{aligned}$$

is applied (see Figure 5.17).

Proof. We calculate the final state of the system O'. This is done according to the P-CTC model [171]. The state $\rho'_{O'}$ is (before renormalization)

$$\rho'_{O'} = \left(\mathbb{1}_{O'} \otimes \langle\Phi|_{O,C}\right)\mathbb{B}\left[\left(U_{I',I} \otimes \mathbb{1}_C\right)\left(\rho_{I'} \otimes |\Phi\rangle\langle\Phi|_{I,C}\right)\left(U^{\dagger}_{I',I} \otimes \mathbb{1}_C\right)\right]\left(\mathbb{1}_{O'} \otimes |\Phi\rangle_{O,C}\right),$$

where $|\Phi\rangle$ is the maximally entangled state entangling the systems I, C, and respectively, O, C. The output of the entanglement breaking channel $\mathbb{B}$ is $\sigma_O \otimes \sigma_{O',C}$ with

$$\begin{aligned}\sigma_O &= \operatorname{Tr}_{O',C}\left(\left(U_{I',I} \otimes \mathbb{1}_C\right)\left(\rho_{I'} \otimes |\Phi\rangle\langle\Phi|_{I,C}\right)\left(U^{\dagger}_{I',I} \otimes \mathbb{1}_C\right)\right)\\ &\propto \operatorname{Tr}_{O'}\left(U_{I',I}\left(\rho_{I'} \otimes \mathbb{1}_I\right)U^{\dagger}_{I',I}\right).\end{aligned} \tag{5.6}$$

Thus, we get

$$\begin{aligned}\rho'_{O'} &= \left(\mathbb{1}_{O'} \otimes \langle\Phi|_{O,C}\right)\left(\sigma_O \otimes \sigma_{O',C}\right)\left(\mathbb{1}_{O'} \otimes |\Phi\rangle_{O,C}\right)\\ &\propto \sum_{i,j}\left(\mathbb{1}_{O'} \otimes \langle i,i|_{O,C}\right)\left(\sigma_O \otimes \sigma_{O',C}\right)\left(\mathbb{1}_{O'} \otimes |j,j\rangle_{O,C}\right)\\ &= \sum_{i,j}\langle i|\sigma_O|j\rangle\left(\mathbb{1}_{O'} \otimes \langle i|_C\right)\sigma_{O',C}\left(\mathbb{1}_{O'} \otimes |j\rangle_C\right).\end{aligned}$$

If we plug in $\sigma_{O',C}$, we get

$$\begin{aligned}
\rho'_{O'} &\propto \sum_{i,j} \langle i|\sigma_O|j\rangle \operatorname{Tr}_O \left((U \otimes \langle i|_C) \left(\rho_{I'} \otimes |\Phi\rangle\langle\Phi|_{I,C} \right) \left(U^\dagger \otimes |j\rangle_C \right) \right) \\
&= \sum_{i,j,k,\ell} \langle i|\sigma_O|j\rangle \operatorname{Tr}_O \left((U \otimes \langle i|_C) \left(\rho_{I'} \otimes |k,k\rangle\langle \ell,\ell|_{I,C} \right) \left(U^\dagger \otimes |j\rangle_C \right) \right) \\
&= \sum_{i,j,k,\ell} \langle i|\sigma_O|j\rangle \operatorname{Tr}_O \left(U \left(\rho_{I'} \otimes |k\rangle\langle \ell|_I U^\dagger \right) \langle i|k\rangle\langle \ell|j\rangle \right) .
\end{aligned}$$

From the last line, it follows that in the sum we have the identities $i = k$ and $j = \ell$. So, we obtain

$$\rho'_{O'} \propto \sum_{i,j} \langle i|\sigma_O|j\rangle \operatorname{Tr}_O \left(U \left(\rho_{I'} \otimes |i\rangle\langle j|_I \right) U^\dagger \right) .$$

Now, we simply shift the expression $\langle i|\sigma_O|j\rangle$ in-between the $|i\rangle\langle j|$ and get

$$\rho'_{O'} \propto \operatorname{Tr}_O \left(U \left(\rho_{I'} \otimes \sigma_O \right) U^\dagger \right) .$$

This last equation is the same as the one used for the D-CTC model (see Equation (5.3)) with the fixed point σ_O (see Equation (5.6)). □

5.6 Discussion

Closed time-like curves (CTCs) are world lines where, if objects would travel on them, then they would bump into themselves. By allowing objects to influence their younger selfs, CTCs become eligible to the grandfather antinomy (see Section 1.2 and Section 3.3.1). Surprisingly, space-time geometries that are consistent with general relativity that *contain* CTCs exist. How should one interpret this discovery? Should we conjecture that "[t]he laws of physics do not allow the appearance of closed time-like curves," just as Hawking [149] did, and see CTCs as a *mathematical artefact*? Or should we consider CTCs as physically *possible* as long as they do not lead to contradictions?

The main part of this chapter consists in studding CTCs under a set of natural principles that one wishes to uphold. The main principle is *Novikov's principle of self-consistency*. This principle asserts that only *self-consistent* dynamics is possible. As argued by Novikov [190] himself, CTCs do not conflict this principle. Another principle is the *no-new-physics* principle. The *weak* form of this principle asserts that physics at some space-like region should remain invariant under the existence or absence of CTCs in the future of that region. Echeverria, Klinkhammer, and Thorne [97] gave strong evidence that this principle could *always* be uphold. Here, we go beyond the studies of Echeverria *et al.* [97] and ask to uphold the *strong* version of the *no-new-physics* principle. In its strong version, this principle asks for physics within local regions to remain invariant no matter whether CTCs exist *around* these regions or not. Thus, local regions are allowed to influence the systems traveling on CTCs in an arbitrary way — as if they would not travel on any CTC. Our main contribution is that these principles do not rule out CTCs. Furthermore, these principles imply that the Cauchy problem is well-defined (see Principle 3), and that every such CTC can be embedded into a larger dynamics that is *reversible* (see Principle 4). These results put CTCs in a different light: CTCs do not only arise as solutions to Einstein's equations of general relativity, but they are reversible and respect a series of natural assumptions. By this, we showed that the existence of CTC in nature *seems less improbable* as previously believed. Furthermore, one does not need to consider quantum dynamics in order to uphold consistency.

Additionally, we showed how our model of CTCs differs from other formulations. The main difference is that in contrast to the other models, our model is *linear*. Nonlinearity leads to many peculiarities (see References [9, 127, 204] for *some* peculiarities), not present in our model.

Chapter 6

Self-referential model of computation

In the previous two sections, we studied loops in terms of *non-causal correlations* and *closed time-like curves*. We saw that in both cases, loops exist that are *unproblematic* in the sense of the assumptions placed in Section 4.1 for the former, and in the sense of the physical and logical principles described in Section 5.2 for the latter. Here, we extend these studies to *computation*: What computations become possible if we introduce *loops*, and what is the *computational power* of such a model? The answer to the second questions is used to make statements about the *physicality* of closed time-like curves (CTCs); the existence of CTCs in nature, where the CTCs behave as described in the previous chapter, is *less implausible* when compared to the commonly used models for CTCs (more on this in the following section).

Traditional models of computation like the Turing machine [238, 239], the circuit model, or the billiard computer [120] are designed to have a linear, *i.e.* causal, time-flow. This means that fundamental operations are carried out *one after the other* (or in parallel). Some programs, then again, are designed in such a way that the output of some system is fed back. An example where feed-back loops are present is an autopilot system of an aircraft. There, some factors like the altitude are measured, after which the power setting and the rudder for instance are readjusted, which then again has an influence on the altitude of the aircraft. This feed-back process continues until the autopilot is turned off. However, such feed-back loops differ qualitatively from causal loops. In the latter, an "effect is the cause of its own cause" while in the former, the cause-effect chain *grows*: The adjusted altitude is *not* the cause for the described change of the aircraft settings, but it is the cause for the *upcoming* change of the settings.

Causal models of computation can be described by Markov chains (see Figure 6.1). By this we mean the following. The state of the computational device at time t can be described by ρ_t. The successor state, *i.e.*, the state *after* ρ_t is ρ_{t+1} *etc.* Furthermore, every state depends on the previous state solely, *i.e.*, ρ_{t+1} depends on ρ_t only. In

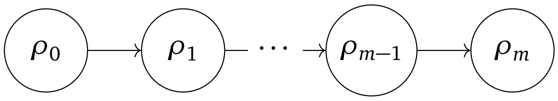


Figure 6.1. Traditional computation can be modeled by a Markov chain where a state ρ_{t+1} depends on the *previous* state ρ_t only.

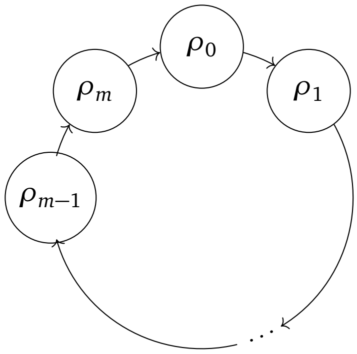


Figure 6.2. Self-referential computation can be described by a Markov chain that is closed; by a cycle where "future" events can influence "past" ones.

stark contrast, we understand a *self-referential* model of computation as a model where "future" states potentially can influence "past" ones. When we put this into the Markovian picture, we can say that a program on a self-referential model can be described by Markov "bracelets" or cycles (see Figure 6.2). Here, the state ρ_{t+1} depends on the state ρ_t — just as for causal circuit —, but additionally, ρ_0, *i.e.*, the "initial" state, depends on ρ_m, *i.e.*, the "final" state. Note that here the terms "initial" and "final" are ambiguous. Basically, for every pair of neighbouring states the earlier could be called "final" and the later "initial."

In terms of a circuit model that consists of gates and wires, a *self-referential circuit* is a circuit where some outputs of same gates are connected to their inputs (potentially through other gates). Put differently, a circuit can be understood as a graph where the gates are nodes and the wires are directed edges. We call such a graph the *graph associated to the circuit*. A *self-referential circuit* is a circuit where its associated graph contains cycles.

6.1 From complexity theoretic considerations to closed time-like curves

We start by some considerations on complexity theory. The class P is the set of all languages that are decidable by a *deterministic* Turing machine in polynomial time. Formally, a language $\mathcal{L} \subseteq \Sigma^*$ is a set of words over some alphabet Σ, and a word x

is a string $x \in \Sigma^*$ of arbitrary length. A language $\mathcal{L}$ is said to be in P if there exists a *deterministic* Turing machine that takes some input $x \in \Sigma^*$ and *decides membership*, *i.e.*, it answers "yes" whenever $x \in \mathcal{L}$ holds, in a running time that is polynomial in the *length* of the input: $|x|$. The class NP, then again, is similar to the class P with the sole difference that a *nondeterministic* Turing machine is considered, as oppose to a *deterministic* one. A *nondeterministic* Turing machine is a Turing machine that runs *all paths of computation* simultaneously. More intuitively, a language $\mathcal{L} \subseteq \Sigma^*$ is said to be in the class NP if there exists a *deterministic* Turing machine such that for every $x \in \mathcal{L}$ there exists a *witness* $w \in \Sigma^*$, and if x and w are used as input to the *deterministic* Turing machine, then it outputs "yes" in a running time that is polynomial in $|x|$. Thus, P is the class of all decision problems that can be *solved efficiently*, *i.e.*, in polynomial time, and the class NP consists of all decision problems where membership can be *verified efficiently*.

Among computer scientists and mathematicians it is believed that $\mathsf{P} \neq \mathsf{NP}$; there exist problems where a solution can be verified easily, yet it is hard to come up with the solution (see the blog post by Aaronson [4] for an overview over P *versus* NP). If $\mathsf{P} = \mathsf{NP}$, then this

> "hätte [...] Folgerungen von der grössten Tragweite. Es würde nämlich offenbar bedeuten, dass man trotz der Unlösbarkeit des Entscheidungsproblems die Denkarbeit des Mathematikers bei ja-oder-nein Fragen vollständig durch Maschinen ersetzen könnte"[1] [137].

This furthermore suggests the so-called "NP-hardness Assumption" [1].

Conjecture 1. No physically realizable model of computation *can solve* NP-hard *problems efficiently.*

This conjecture allows to make statements about the *physical realizability* of closed time-like curves (CTCs). Such statements are obtained by studying the computational power of a computational model that *incorporates CTCs*, *e.g.*, a model that has access to CTCs. If such a model can solve NP-hard problems efficiently, then closed time-like curves are *unphysical* under this conjecture.

Deutsch's [93] model of closed time-like curves (D-CTC) (see Section 5.4.1) was also examined in terms of computation. As Aaronson and Watrous [6] showed, this model can solve PSPACE-complete problems in polynomial time. Aaronson [3] gave some intuition for that result. The class PSPACE is the class of problems solvable by a polynomial amount of space. Now, since D-CTC can solve the same problems efficiently in time, and since with CTCs one can go back in time, the D-CTC model renders *time reusable*, just as space is.

[1] As translated in Reference [137]: "would have consequences of the greatest significance. Namely, this would clearly mean that the thinking of a mathematician in the case of yes-or-no questions could be completely replaced by machines, in spite of the unsolvability of the Entschidungsproblem."

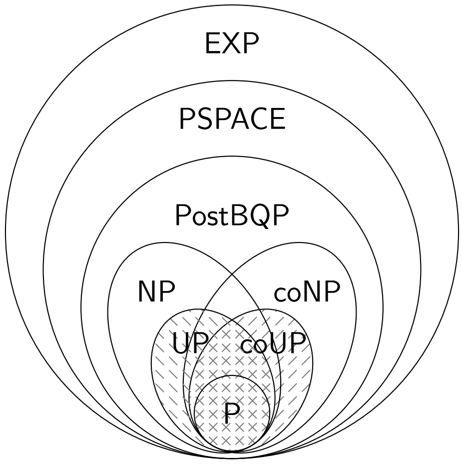


Figure 6.3. Schematic representation of some complexity classes. Our focus in the later part of this chapter is on the intersection between UP and coUP (marked with crosses).

Even more strikingly, the same is shown for the classical variant of D-CTC. Let us denote by $\mathsf{BQP}_{\mathrm{DCTC}}$ the class of decision problems efficiently, *i.e.*, in polynomial-time, solvable with a *quantum* D-CTC, and likewise $\mathsf{P}_{\mathrm{DCTC}}$ for the classical special case of D-CTC.[2] So, Aaronson and Watrous [6] showed $\mathsf{P}_{\mathrm{DCTC}} = \mathsf{BQP}_{\mathrm{DCTC}} = \mathsf{PSPACE}$. Most recently, Aaronson *et al.* [7] proved that the D-CTC model can solve the halting problem.

The other widely used model for CTCs is based on post-selection and teleportation (P-CTC) (see Section 5.4.2). Also here, the computational power of P-CTC was studied. Let $\mathsf{BQP}_{\mathrm{PCTC}}$ be the class of all decision problems efficiently solvable in the P-CTC model, and let $\mathsf{P}_{\mathrm{PCTC}}$ be the classical counter part. It was shown [2, 170, 171] that

$$\mathsf{BQP}_{\mathrm{PCTC}} = \mathsf{PostBQP} = \mathsf{PP}\,, \quad \text{and}$$
$$\mathsf{P}_{\mathrm{PCTC}} = \mathsf{BPP}_{\mathrm{path}}\,.$$

The class $\mathsf{BPP}_{\mathrm{path}}$ is the classical analogue of PostBQP [144]. By the relations (see also Figure 6.3)

$$\mathsf{NP} \subseteq BPP_{\mathrm{path}} \subseteq \mathsf{PostBQP} \subseteq \mathsf{PSPACE} \subseteq \mathsf{EXP}\,,$$

the P-CTC model is believed to be computationally *weaker* than D-CTC.

Thus, the D-CTC and the P-CTC models allow for efficient computation of problems that are NP-hard — which is at odds with the conjecture above. This renders both, the D-CTC and P-CTC model, *unphysical*. In contrast to these results, we show that

[2] Note that the complexity zoo [5] refers to these classes as $\mathsf{P}_{\mathrm{CTC}}$ and $\mathsf{BQP}_{\mathrm{CTC}}$ as opposed to $\mathsf{P}_{\mathrm{DCTC}}$ and $\mathsf{BQP}_{\mathrm{DCTC}}$ as defined here. We nevertheless stick to this latter notation to make it for the reader easier to distinguish these classes from the P-CTC classes.

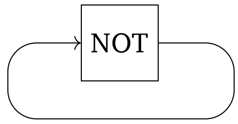


Figure 6.4. This circuit is overdetermined: No consistent assignment of a bit to the wire exist.

the CTC model presented in Chapter 5 is not at all that powerful. Rather, the class of decision problems efficiently solvable by such CTCs is upper bounded by $\mathsf{UP} \cap \mathsf{coUP}$, which is believed to be *strictly* contained within NP. In formulas, if $\mathsf{P}_{\mathsf{CTC}}$ is the class of all decision problems solvable in polynomial time with a Turing machine augmented with the CTCs from Chapter 5, then

$$\mathsf{P}_{\mathsf{CTC}} \subseteq \mathsf{UP} \cap \mathsf{coUP} \,. \tag{6.1}$$

The class UP [241] (see also [197]) is similar to NP with the difference that the problems are restricted to have a *unique* proof for membership. One of our main technical results of this chapter is that

$$\mathsf{P}_{\mathsf{SelfRef}} = \mathsf{UP} \cap \mathsf{coUP} \,,$$

i.e., the computational power of *self-referential circuits* is characterized by $\mathsf{UP} \cap \mathsf{coUP}$. The upper bound in Equation (6.1) follows from the fact that the self-referential model uses a *subset* of the principles from Section 5.2. Therefore, our model of CTCs is *not ruled out by computer-science arguments*. By this, the existence of such CTCs in nature seems *less implausible* than D-CTCs or P-CTCs. At this point, and with the last main chapter of this thesis, we arrive at the conclusion that we *cannot rule out CTCs by logical, physical, or computational arguments*. Let us now construct the tools to arrive at this conclusion.

6.2 Under- and overdetermination

There are two main problems when we connect some output of a gate to itself. These problems are in fact *antinomies*: logical problems (see Section 1.2 and Section 3.3.1). The first problem is the problem of *overdetermination* — see also the grandfather antinomy. A circuit is said to be overdetermined if the bits on the wires take *contradictory* values. An example of such a circuit is the NOT-gate on bits, where the output is connected to the input (see Figure 6.4). The NOT-gate describes the following function

$$\begin{aligned} \mathrm{NOT} : \{0,1\} &\to \{0,1\} \\ a &\mapsto b = \neg a \,. \end{aligned}$$

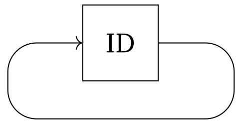


Figure 6.5. This circuit is underdetermined: Both bits 0 and 1 are possible; the circuit does not specify the bit on the wire.

If the *output* of that gate is connected to its input, we get the additional identity

$$a = b\,.$$

Both equations together form a system of equations that is *overdetermined*.

The other problem is the problem of *underdetermination*, which is essentially the information antinomy. In such a case, the bits on the wires are *not defined* by the circuit. This is illustrated by the circuit which consists of the identity gate ID, and where the output is connected to the input (see Figure 6.5). Here, the gate describes the function

$$\begin{aligned} \mathrm{ID} : \{0,1\} &\to \{0,1\} \\ a &\mapsto b = a\,, \end{aligned}$$

and by connecting the output with the input we additionally require for

$$a = b\,.$$

Such a system of equations is *underdetermined*: The variable a is undefined. The problem with underdetermination seems slightly more subtle than overdetermination. The latter is not possible because otherwise 0 must be equal 1 — a contradiction. The former, then again, does *not compute the output*; the output of the circuit, or the bits on the wires, cannot be specified. Deutsch [93] refers to the information antinomy as "creationism:" The bits are not specified by the model but, if that would be possible, would arise *ex nihilo*.[3]

Definition 19 (Logically consistency of circuits)**.** In the spirit of the previous two chapters, we call a circuit *logically consistent* if and only if the bits on the wires are *not overdetermined* and *not underdetermined*.

This definition does not restrict the circuits to be *causal*, and in fact, we show that this assumption of logical consistency is *strictly weaker* than the assumption of a linear time-flow. What follows from this definition is that a *self-referential* circuit is logically consistent if it has a *unique* fixed point on the "looping" wires. No overdetermination is the same as saying there is *at least one* fixed point. Then again, no underdetermination asks for *not more than one* fixed point.

[3]The full quote of Deutsch [93] reads: "It is a fundamental principle of the philosophy of science that the solutions of problems do not spring fully formed into the Universe, *i.e.*, as initial data, but emerge only through evolutionary or rational processes. In adopting this *evolutionary principle* we reject such antirational doctrines as creationism [. . .]."

6.2.1 Put in context with closed time-like curves and non-causal correlations

We briefly compare the requirements of no under- and no overdetermination with the assumptions used for the non-causal correlations, and with the principles used in the discussion of the CTCs.

Our CTCs were built up on four principles (see Section 5.2). These were Novikov's principle of self-consistency, the no-new-physics principle, the uniqueness principle, and reversible dynamics (actually, the latter two principles follow from the former two). Novikov's principle asserts that *consistent* dynamics arise only. This is basically the same as requiring that *no overdetermination* takes place. The no-new-physics principle asks for all operations to be possible within a local space-time region. This assumption is dropped for the model of computation presented here. We drop this assumption because we do not incorporate the notion of parties that can freely choose their operations within a local region. Rather, local regions are replaced by *fixed* gates, and the term *logical consistency* applies to the circuit as a whole. The *uniqueness principle* can be seen as following from the assumption of *no over- and no underdetermination*. And finally, we do not make any assumptions on reversibility in the modeled studied here.

Thus, this self-referential model of computational uses a subset of the principles for the closed time-like curves. This means that every closed time-like curve obeying the principles from Section 5.2 can be transformed into such a circuit. Physically, one can think of some space-like surface $\mathcal{P}$ where the circuit is designed. Now, if in the future of $\mathcal{P}$ there exists a CTC (as in Figure 5.2), then this designed circuit can implemented on that CTC where it is evaluated.

When we compare this model of computation with the framework for non-causal correlations, then we see that the model for correlations could in some cases be *more powerful*. The reason for this is that the framework allows for processes that are *intrinsically probabilistic* — they cannot be expressed as convex combinations of *deterministic* ones. Therefore, not every setup in the framework for non-causal correlations can be translated into a self-referential circuit. This has the consequence that $\mathsf{P}_{\text{SelfRef}}$ is not necessarily an upper bound on the computational power of non-causal correlations. The so-called *fine-tuned* environments (see the polytope presented in Example 6) might have a higher power. This is not examined and is left as an open question.

6.2.2 Put in context with Gödel's incompleteness theorems

Under- and overdetermination can also put in a different light. Gödel [130] showed in his seminal work that "*every* formal system containing arithmetic" [136] cannot be *complete and consistent* at the same time. If a formal system contains sentence that can be proven within that system to be *true as well as false*, then the system is called *inconsistent*. Then again, if the formal system contains sentence that cannot be proven *neither true nor false*, then such a system is *incomplete*. *Overdetermination* is similar to *inconsistency* in the Gödel sense; some bit on a wire is *both*, 0 and 1. *Underdetermination* is similar to *incompleteness* in the Gödel sense; there exist bits that are *not specified*.

6.2.3 Put in context with anthropic computing

These considerations on under- and overdetermination already give us a hint on how such circuits might be programed. Suppose you are given a *causal* circuit that produces a guess for a solution to some problem. At this stage of discussion is seems as if an *incorrect* guess (let us assume we can verify the guess) can be eliminated by introducing an overdetermination — a logical contradiction. This is very much like guessing the solution to a problem and killing ones own grandfather if a wrong guess was produced; a computational model termed "anthropic computing" [1] or "quantum suicide" [232]. Aaronson proposed the name "anthropic computing" in analogy to the *anthropic principle* in cosmology: The probability of being alive depends on the guess. In that model, one could solve NP-complete problems in polynomial time. In DeWitt's [95] and Deutsch's [94] interpretation of the Hermann-Everett interpretation [111, 152] of quantum theory, a quantum measurement leads to a *branching* of the universe, where every possible measurement result occurs in some universe. If our guess is subject to *quantum randomness*, then we either end up in a universe where the guess was correct or where the guess was wrong. In the case we end up in a branch where the guess is correct, we solved the problem. However, if we end up in a branch with a *wrong* guess, then we could just commit suicide (please do not try this!) — and by this stop our conciousness to persist in that branch. By that, our conciousness ends up in those branches only where the guess was correct. However, this requires that we can verify the guess and, additionally, that conciousness persists in the other branches after the suicide in one branch. One very sad "quantum suicide" was in fact carried out by Everett's daughter: "In her suicide note she wrote that she was going off to meet her father in a parallel universe" [172].

6.3 Model

The self-referential model of computation is inspired by the framework of correlations without causal order (see Chapter 4). There, we studied the correlations arising when parties are not subject to a *a priori* causal order. Furthermore, the parties are allowed to perform *arbitrary* operations, and we asked for the probabilities to be well-defined for any choice or operations. The latter led to the notion of environment which connects the parties in one or another way. Here, in contrast, this notion of parties and environment is replaced by *gates*. Gates, then again, are not experimenters that can freely choose their interactions; gates are fixed. A circuit in this model consists of *gates* and *wires* connecting these gates. As said before, the gates can be connected *arbitrarily* as long as the circuit as a whole is logically consistent.

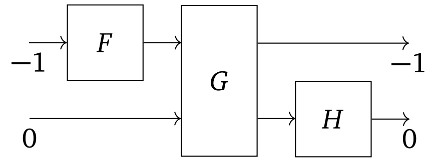


Figure 6.6. An example of a causal circuit with two inputs and two outputs.

Let us state some definition that should make the discussion more precise. We can always built up a self-referential circuit by starting from a causal one.

Definition 20 (Causal circuit)**.** A *causal circuit* $\mathcal{C}$ is a circuit that consists of gates and wires connecting these gates, where furthermore the associated graph is *acyclic*. Without loss of generality, and if not otherwise stated, we assume that the wires carry bits, and that number of input wires equals the number of output wires. Additionally, all input wires are labeled with successive integers (possibly negative ones as well). The output wires, then again, are labeled with the same integers. Such a circuit can, furthermore, be identified with a function (we use lowercase letters to denote these functions)

$$c : \mathcal{A} \times \mathcal{X} \to \mathcal{A} \times \mathcal{X} ,$$

where the set $\mathcal{A}$ is the set of possible inputs to the wires with *non-negative* labels, and the set $\mathcal{X}$ is the set of possible inputs to the negatively labeled wires. The negatively labeled wires are transformed to loops in a later part of this work. Since the number of input and output wires are the same, c maps $\mathcal{A} \times \mathcal{X}$ to itself.

An example of a causal circuit is given in Figure 6.6. A *self-referential* circuit $\mathcal{C}'$ is constructed from a causal circuit $\mathcal{C}$.

Definition 21 (Self-referential circuit)**.** Given a causal circuit $\mathcal{C}$, the *self-referential circuit* $\mathcal{C}'$ is obtained by connecting the output wires with the input wires that have the same *negative* label. Furthermore, we add a "read-out" wire that carries as many bits k as the negatively labeled output wires of $\mathcal{C}$ and where the state c on those wires is added to the "read-out" wire modulo k. The "read-out" wire is labeled with the successive positive integer.

We adopt Definition 19 of logical consistency.

Definition 22 (Logically consistent circuit)**.** From the definition of logical consistency above, a self-referential circuit $\mathcal{C}'$ constructed from a causal circuit $\mathcal{C}$ is logically consistent if and only if

$$\forall a \in \mathcal{A}, \exists!(a', x) \in \mathcal{A} \times \mathcal{X} : c(a, x) = (a', x) .$$

Here, the variable x represents the value on the looping wires, the variable a is the input, and the variable a' is the output from $\mathcal{C}'$.

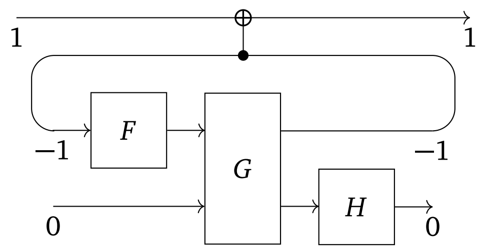


Figure 6.7. An example of a self-referential circuit built from the causal one depicted in Figure 6.6.

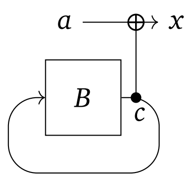


Figure 6.8. The black box B is guaranteed to have a unique fixed point. This fixed point is found by transforming B to a closed circuit and by reading out the value on the wire.

As an example, the circuit shown in Figure 6.7 is logically consistent if for every input on the wire labeled 0 and 1, the "looping" wire (label -1) has a unique fixed point.

6.4 Query complexity

Suppose you are given some black box B that takes one input $a \in \mathcal{B}$ and produces one output $x \in \mathcal{B}$. Furthermore, you are guaranteed that the black box B has a *unique* fixed point. In the traditional circuit model, $|\mathcal{B}|-1$ queries to the black box are needed in the worst case to find the fixed point. Now, the self-referential model described above can give us the fixed point by *one* query. This is shown in Figure 6.8. One might question whether indeed the black box is queried *only once*. But this can be seen to be the case for the following reason: Any other query would be *inconsistent*.

But how should we proceed if we are given a black box B_2 where we are guaranteed that it has *two* fixed points, c and c'? The above recipe to find the unique fixed point of a black box *fails*. The problem is that since we have two fixed points, the self-referential circuit B_2' constructed from the causal one B_2 is *underdetermined* — both fixed points could potentially travel on the looping wire. However, by some trick similar to "quantum suicide," it is possible to find these fixed points in two queries. This is shown in Figure 6.9. There, the gate G acts in the following way:

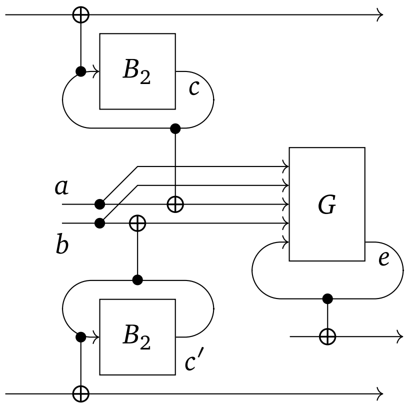


Figure 6.9. The black box B_2 is guaranteed to have two fixed points. These are found by independently transforming the black boxes to closed circuits, and then by enforcing that the values on the closed circuits are *different* and *ordered*. This ensures a unique fixed point of the whole circuit.

$$
\begin{aligned}
g : \mathcal{B}^4 \times \{0,1\} &\to \{0,1\} \\
(a, b, a', b', e) &\mapsto \begin{cases} 0 & \text{if } a' \ominus a < b' \ominus b\,, \\ e \oplus 1 & \text{otherwise,} \end{cases}
\end{aligned}
$$

where $\ominus$ is subtraction modulo $|\mathcal{B}|$, and where $\oplus$ is addition modulo 2. If the fixed points obtained from the two black boxes are the same or ordered decreasingly, then the gate G introduced a contradiction — "killing the grandfather" — by flipping the bit e. Otherwise, if the fixed points are different and given in the increasing order, then the gate outputs 0. By this, the only fixed point that exists on all looping wires is such that the above loop carries the smaller fixed point, the bottom loop carries the larger fixed point, and the loop connected to G carries 0.

6.5 Computational complexity

A *decision problem* Π is often cast as the membership problem of a language $\mathcal{L} \subseteq \Sigma^*$ with alphabet Σ. For simplicity, and without loss of generality, we choose $\Sigma = \{0,1\}$. An instance of a decision problem Π is a string $x \in \Sigma^*$, and the question is: Is x a word of $\mathcal{L}$, *i.e.*, does $x \in \mathcal{L}$ hold? An algorithm that solves a decision problem outputs either "yes" or "no."

Search problems, then again, are mostly defined via binary relations. Such a problem Π is associated with a binary relation $R \subseteq \Sigma^* \times \Sigma^*$. An instance of Π in that case is some $x \in \Sigma^*$, and the question is: *What* (if there exists one) is $y \in \Sigma^*$ such that $(x, y) \in R$? An algorithm that solves a search problem Π outputs y if there exists a y satisfying $(x, y) \in R$, and returns "no" otherwise.

By the expression $|x|$ we denote the length of some string $x \in \Sigma^*$. For the upcoming definitions, we need to introduce the following two notions: *polynomial decidability* and *polynomial balancedness*. A binary relation R is called *polynomially decidable* if there exists a deterministic Turing machine deciding the language $\{(x,y) \in R\}$ in polynomial time. In other words, by this we mean that there exists a deterministic algorithm that solves the decision problem for the language $\mathcal{L} = \{(x,y) \in R\}$. Then again, a binary relation R is called *polynomially balanced* if there exists a polynomial q such that $(x,y) \in R$ implies $|y| \leq q(|x|)$. This notion ensures that the length of the solution to an instance x of a search problem is polynomially bounded by the length of the instance.

In the following definitions, we assume that for every problem Π and for every string $x \in \Sigma^*$, we can check in polynomial time whether x is an instance of Π or not. Note that x might be mall-formed. In the case where x is *not* an instance of Π, then we abort. An introduction to common concepts in complexity theory — as those just stated — can be found in the textbooks by Papadimitriou [197] and by Arora and Barak [18].

We now define the algorithm that takes as input some bit string x and, based on that, generates a self-referential circuit. In this complexity theoretic setup, we have to restrict ourselves to a set of basic gates from which the circuits are constructed. Otherwise it would be impossible to study complexity classes; one could then just define the gates to implement some function that could, therefore, be regarded as black box. The basic blocks the following circuits are constructed from are the AND, the OR, the NOT, and the FAN-OUT (copy) gate. These four gates are *universal* in terms of computability.

Definition 23 (Deterministic SRefCirc algorithm)**.** A *deterministic SRefCirc algorithm* $\mathcal{A}$ is a polynomial-time deterministic algorithm that takes as input a bit string $x \in \{0,1\}^*$ and outputs a Boolean causal circuit $\mathcal{C}_x$ over AND, OR, NOT, and FAN-OUT, such that the self-referential circuit $\mathcal{C}'_x$ is logically consistent, *i.e.*,

$$\exists! y : c_x(y) = y \, .$$

If the first bit of the fixed point y is a 1, *i.e.*, if $y = 1z$ for some z, then we say that $\mathcal{A}$ *accepts* x, otherwise, $\mathcal{A}$ *rejects* x. The algorithm $\mathcal{A}$ *decides a language* $\mathcal{L} \subseteq \{0,1\}^*$ if $\mathcal{A}$ accepts every $x \in \mathcal{L}$ and rejects every $x \notin \mathcal{L}$. Furthermore, the algorithm $\mathcal{A}$ *decides a binary relation* $R \subseteq \{0,1\}^* \times \{0,1\}^*$ if for every $x \in \{0,1\}^*$ the pair (x,y), with $c_x(y) = y$, is in R.

This definition can now be used to define the two complexity classes $\mathsf{P}_{\text{SRefCirc}}$ and $\mathsf{FP}_{\text{SRefCirc}}$. The former complexity class is a class of *decision* problems, and the latter a class of *search* problems. Our main focus is then to characterize these classes in terms of other classes. In particular, we are interested in the separation of these classes from the class NP and from the class FNP (see Section 6.1).

Definition 24 ($\mathsf{P}_{\text{SRefCirc}}$ and $\mathsf{FP}_{\text{SRefCirc}}$)**.** The class $\mathsf{P}_{\text{SRefCirc}}$ contains every *language* that is decidable by any deterministic SRefCirc algorithm. The class $\mathsf{FP}_{\text{SRefCirc}}$ contains every *binary relation* that is decidable by any deterministic SRefCirc algorithm.

As it turns out, we can relate these classes to UP and to the complement thereof.

Definition 25 (UP and coUP)**.** The class UP (Unambiguous Polynomial-time) contains all languages $\mathcal{L}$ for which a polynomial-time verifier $V : \{0,1\}^* \to \{0,1\}$ exists such that for every x, if $x \in \mathcal{L}$ then $\exists! y : V(x,y) = 1$, and if $x \notin \mathcal{L}$ then $\forall y : V(x,y) = 0$. The class coUP consists of all languages $\mathcal{L}$ where the complement of $\mathcal{L}$ is in UP.

The complexity class UP was first defined by Valiant [241]. It basically contains all search problems where *membership* (*non-membership* for coUP) of x in $\mathcal{L}$ can be proven efficiently by a *unique* witness y. Note the close connection between the class UP and NP: The latter class is the same as the former with the sole difference that *multiple* witnesses for membership are allowed in NP. So, NP is defined as UP where the *uniqueness* quantifier is replaced by an *existence* quantifier.

In the following theorem, we show that $\mathsf{P}_{\mathrm{SRefCirc}}$ equals $\mathsf{UP} \cap \mathsf{coUP}$. Known problems in $\mathsf{UP} \cap \mathsf{coUP}$ are integer factorization (more precisely, the decision problem analogue thereof: Does N has a factor greater than k?) [116] and parity games [161]. In Section 6.6, we design a self-referential circuit that factorizes integers in polynomial time. An implication of the following theorem is that $\mathsf{UP} \cap \mathsf{coUP}$ can be understood in terms of fixed point.

Theorem 17. $\mathsf{P}_{\mathit{SRefCirc}} = \mathsf{UP} \cap \mathsf{coUP}$.

Proof. We prove this identity by showing that the left class in contained in the right and *vice versa*. Let us start with $\mathsf{P}_{\mathrm{SRefCirc}} \supseteq \mathsf{UP} \cap \mathsf{coUP}$. Assume a language $\mathcal{L}$ is in $\mathsf{UP} \cap \mathsf{coUP}$. This means that there exist two polynomial-time verifiers V_{yes} and V_{no} (one for UP and one for coUP) such that for every x, if $x \in \mathcal{L}$, then

$$\exists! w : V_{\mathrm{yes}}(x,w) = 1 \wedge \forall w' : V_{\mathrm{no}}(x,w') = 0\,,$$

and if $x \notin \mathcal{L}$

$$\forall w : V_{\mathrm{yes}}(x,w) = 0 \wedge \exists! w' : V_{\mathrm{no}}(w,x') = 1\,.$$

Now, we construct a deterministic SRefCirc algorithm $\mathcal{A}$ that decides the language $\mathcal{L}$. The algorithm $\mathcal{A}$ takes some $x \in \{0,1\}^*$ and generates the self-referential circuit $\mathcal{C}'_x$ from the causal circuit $\mathcal{C}_x$ shown in Figure 6.10. The subcircuits $\mathcal{V}_{\mathrm{yes}}, \mathcal{V}_{\mathrm{no}}$ implement the verifiers $V_{\mathrm{yes}}, V_{\mathrm{no}}$. Because $\mathcal{L}$ is assumed to be in $\mathsf{UP} \cap \mathsf{coUP}$, we can construct and run these subcircuits in polynomial time. The circuit $\mathcal{C}_x$ acts in the following way:

$$\begin{aligned} c_x : \{0,1\} \times \{0,1\}^{q(|x|)} &\to \{0,1\} \times \{0,1\}^{q(|x|)}\,, \\ (b,w) &\mapsto \begin{cases} (0,w) & V_{\mathrm{no}}(x,w) = 1\,, \\ (1,w) & V_{\mathrm{yes}}(x,w) = 1\,, \\ (b \oplus 1, w) & \text{otherwise,} \end{cases} \end{aligned}$$

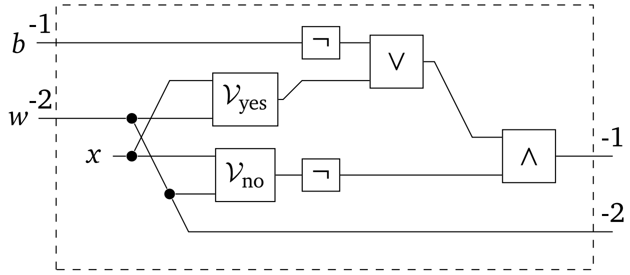


Figure 6.10. This closed circuit constructed from this causal circuit $\mathcal{C}_x$ is used to reduce a problem from UP∩coUP to $\mathsf{P}_{\text{SRefCirc}}$. The wire that carries the witness w for membership or non-membership consists of $q(|x|)$ bits.

where q is a polynomial. This circuit $\mathcal{C}'_x$ is *logically consistent* because it contains a *unique* fixed point. If $x \in \mathcal{L}$, then there exists a unique w with $V_{\text{yes}}(x,w) = 1$, and $c_x(1,w) = (1,w)$. Otherwise, there exists a unique string w with $V_{\text{no}}(x,w) = 1$, and $c_x(0,w) = (0,w)$.

The converse ($\mathsf{P}_{\text{SRefCirc}} \subseteq$ UP∩coUP) holds for the following reason. First, assume $\mathcal{L}$ is in $\mathsf{P}_{\text{SRefCirc}}$. Thus, we have an algorithm $\mathcal{A}$ that, for every x, produces the logically consistent circuit $\mathcal{C}'_x$. We design both verifiers V_{yes} and V_{no} to act as

$$
\begin{aligned}
V_{\text{yes}} &: (x,z) \mapsto c_x(z) = z \wedge z = 1w\,,\\
V_{\text{no}} &: (x,z) \mapsto c_x(z) = z \wedge z = 0w\,.
\end{aligned}
$$

In words, both verifiers check whether z is a fixed point of $\mathcal{C}_x$, and additionally check for the first bit. If z is a fixed point and the first bit of z is a 1, then $x \in \mathcal{L}$. Otherwise, if z is a fixed point and the first bit of z is a 0, then $x \notin \mathcal{L}$. So, for both cases, membership and non-membership, a unique witness exists that is polynomial in the size of the instance, and by that $\mathcal{L} \in$ UP∩coUP. □

So much for decision problems. An analogous theorem for search problems can be stated as well. For that, we introduce the *functional* variants of the complexity classes discussed before.

Definition 26 (FUP)**.** A binary relation R is in FUP (Function UP) if and only if R is polynomially decidable, polynomially balanced, and $\forall x : |\{y \mid (x,y) \in R\}| \leq 1$.

Informally, a problem is in FUP if for every instance there exists *at most* one solution.

Definition 27 (F(UP∩coUP))**.** A pair (R_1, R_2) of relations is in F(UP∩coUP) if and only if both relations are polynomially decidable, polynomially balanced, and for every instance x

$$
\begin{aligned}
&(\exists! y : (x,y) \in R_1 \wedge \forall z : (x,z) \notin R_2) \oplus\\
&(\forall y : (x,y) \notin R_1 \wedge \exists! z : (x,z) \in R_2)
\end{aligned}
$$

holds. The exclusive-or ($\oplus$) asks for *either* yet *not both* expressions to be true.

Note that the output of a search problem in $\mathsf{F}(\mathsf{UP} \cap \mathsf{coUP})$ is some string w that satisfies either $(x,w) \in R_1$ or $(x,w) \in R_2$ but not both. Furthermore, as we formulated it, the string w does not indicate in *which* relation the pair (x,y) appears. Since both relations are polynomially decidable and polynomially balanced, however, we can check in polynomial time whether y is a solution of R_1 or R_2. This brings us to the following class, which is equal to $\mathsf{F}(\mathsf{UP} \cap \mathsf{coUP})$.

Definition 28 (TFUP)**.** A binary relation R is in TFUP (Totally FUP) if and only if R is polynomially decidable, polynomially balanced, and $\forall x, \exists! y : (x,y) \in R$.

The class TFUP is the analog to $\mathsf{TFNP} = \mathsf{F}(\mathsf{NP} \cap \mathsf{coNP})$ [178]. This is the class of *total* relations that are polynomially decidable and polynomially balanced.

Theorem 18. $\mathsf{TFUP} = \mathsf{F}(\mathsf{UP} \cap \mathsf{coUP})$.

Proof. Let R be a relation in TFUP and R_1, R_2 two relations such that for every x:

$$
\begin{aligned}
&(\exists! y : (x,y) \in R_1 \wedge \forall z : (x,z) \notin R_2) \oplus \\
&(\forall y : (x,y) \notin R_1 \wedge \exists! z : (x,z) \in R_2)\ .
\end{aligned}
$$

The inclusion $\mathsf{TFUP} \subseteq \mathsf{F}(\mathsf{UP} \cap \mathsf{coUP})$ can be seen by setting $R_1 = R$ and $R_2 = \emptyset$. The converse is obtained by setting $R = R_1 \cup R_2$. □

We now show that the class of search problems efficiently solvable by self-referential circuits coïncides with TFUP.

Theorem 19. $\mathsf{FP}_{SRefCirc} = \mathsf{TFUP}$.

Proof. We prove this identity in a similar way to Theorem 17. First, we show the inclusion $\mathsf{TFUP} \subseteq \mathsf{FP}_{\mathrm{SRefCirc}}$. By definition, a binary relation R that is in TFUP is polynomially decidable and polynomially balanced. Therefore, there exists an algorithm $\mathcal{D}$ that takes two inputs x, y, runs in polynomial time in $|x|$, and if $(x,y) \in R$ then $\mathcal{D}$ outputs “yes,” otherwise, $\mathcal{D}$ outputs “no.” Furthermore, for every instance x there exists a *unique* y with $(x,y) \in R$. The deterministic SRefCirc algorithm $\mathcal{A}$, upon receiving x, generates the self-referential circuit $\mathcal{C}'_x$ that is constructed from the causal circuit $\mathcal{C}_x$, which then again acts as

$$
c_x : y \mapsto \begin{cases} y & (x,y) \in R\,, \\ y' & \text{otherwise,} \end{cases}
$$

where, if $y = bz$ with $b \in \{0,1\}$, then $y' = (b \oplus 1)z$, *i.e.*, the first bit gets flipped in the case of $(x,y) \notin R$. Therefore, for every x we have a self-referential circuit $\mathcal{C}'_x$ with a unique fixed point (logically consistent). The fixed point, furthermore, is the solution of the search problem: $c_x(y) = y \rightarrow (x, y) \in R$. The converse inclusion relation $\mathsf{FP}_{\text{SRefCirc}} \subseteq \mathsf{TFUP}$ is shown as follows. Suppose we are given a relation R that is decidable by a deterministic SRefCirc algorithm $\mathcal{A}$. We now need to show that R is polynomially decidable, polynomially balanced, and that every instance x has a *unique* solution. Indeed, R is polynomially decidable because $\mathcal{C}_x$ is generated in polynomial time in $|x|$. Furthermore, R is polynomially balanced because the circuit $\mathcal{C}_x$ with input y runs in polynomial time in $|x|$. Finally, $\mathcal{C}_x$ has a *unique* fixed point — it is logically consistent. Thus, the algorithm $\mathcal{D}$ that decides R simply returns "yes" if $c_x(y) = y$ and "no" otherwise. □

6.6 Example: Integer factorization

Integer factorization is a search problem that is in TFUP [116]. Therefore, we are able to construct a self-referential circuit that factorizes some number N in polynomial time. The solution to the search problem is the decomposition of N into a product of prime numbers.

The SRefCirc algorithm $\mathcal{A}$ outputs, on input $N \in \mathbb{Z}$, a circuit $\mathcal{C}_N$ with which the number $N = p_1^{e_1} p_2^{e_2} \dots$ can be decomposed into its prime factors $p_1, p_2, \dots$ along with its exponents $e_1, e_2, \dots$. We give a description of $\mathcal{C}_N$ as an algorithm. Clearly, this algorithm can be transformed into a circuit. The following algorithm runs in a time polynomial in the instance size $n = \lceil \log N \rceil$.

Algorithm 1 Factoring N

Input: $b \in \{0, 1\}, a_1, a_2, \dots, a_n, e_1, e_2, \dots, e_n \in K$
Output: $b' \in \{0, 1\}, a_1, a_2, \dots, a_n, e_1, e_2, \dots, e_n \in K$

$w \leftarrow \neg b, a_1, a_2, \dots, a_n, e_1, e_2, \dots, e_n$
for $i = 1$ to $n - 1$ **do**
  if $(a_i < a_{i+1}) \vee (a_i \neq 1 \wedge a_i = a_{i+1})$ **then**
    return w
  end if
end for
for $i = 1$ to n **do**
  if $(a_i = 1 \wedge e_i > 1) \vee a_i \notin \mathsf{PRIME} \cup \{1\}$ **then**
    return w
  end if
end for
if $a_1^{e_1} a_2^{e_2} \dots a_n^{e_n} \neq N$ **then**
  return w
end if
return $0, a_1, a_2, \dots, a_n, e_1, e_2, \dots, e_n$

Algorithm 1 takes as input 1 bit and $2n$ numbers in $K = \{1, 2, \ldots, N-1\}$, where every number is represented as an n-bit string. On line 3 we check whether the first n numbers are ordered. On line 8 we check whether e_i is 1 whenever $a_i = 1$, and whether a_i is indeed prime (or 1). A deterministic primality test can be performed in polynomial time as was recently shown [10]. Finally, on line 12 we check whether the decomposition is correct. If all tests are passed, then the algorithm returns $0, a_1, a_2, \ldots, a_n, e_1, e_2, \ldots, e_n$ where $\prod_{i=1}^{n} a_i^{e_i} = N$, otherwise, the algorithm *flips* the first input bit. This algorithm and, therefore, the circuit $\mathcal{C}_N$, has a *unique* fixed point

$$0, p_1, p_2, \ldots, p_m, 1^{n-m}, e_1, e_2, \ldots, e_m, 1^{n-m},$$

where $p_1 > p_2 > \cdots > p_m$ are primes and $\prod_{i=1}^{m} p_i^{e_i} = N$.

6.7 Discussion

We designed a circuit model of classical computation where all circuits are *not under- nor overdetermined*. Since we did not assume that a circuit has to be *causal*, a gate can be connected to *itself* (potentially through other gates). This, however introduces “loops” to such circuits. The assumption of *no under- and no underdetermination* then again ensures that the so-called self-referential circuits have unique fixed points. This leads to the following implications.

The assumption of *no under- and no underdetermination* is strictly weaker than the assumption of a causal ordering of the gates. We demonstrated this by showing an advantage in query complexity when compared to causal circuits (see Section 6.4).

Furthermore, we were able to precisely characterize the computational complexity of such circuits. For decision problems, we obtained that $\mathsf{P}_{\text{SRefCirc}} = \mathsf{UP} \cap \mathsf{coUP}$, and for search problems, the analogous identity $\mathsf{FP}_{\text{SRefCirc}} = \mathsf{TFUP}$. The former class is a subset of NP: $\mathsf{UP} \cap \mathsf{coUP} \subseteq \mathsf{NP}$. Furthermore, *strict* inclusion is conjectured because $\mathsf{UP} \cap \mathsf{coUP}$ is not believed to contain any complete problems [227]. This means that the closed time-like curves (CTCs) described in Chapter 5, which use a superset of the assumptions posed here, cannot solve NP-hard problems. If one presupposes the NP-hardness assumption, *i.e.*, no physically realizable model of computation can solve NP-hard problems, then the CTCs from Chapter 5 are *innocent* from a computer-science perspective. This result contrasts the power of other model of CTCs (the D-CTC and the P-CTC models) as they are able to solve NP-complete problems in polynomial time.

A key assumption in current cryptographic schemes is the difficulty of integer factorization. This problem is known to be in $\mathsf{UP} \cap \mathsf{coUP}$ [116], and is thus solvable in polynomial time with our model (see Section 6.6). Yet, as the CTC model uses a strict superset of the assumptions used here, it is not clear whether it is possible to factorize integers with such CTCs.

The identification $\mathsf{P}_{\text{SRefCirc}} = \mathsf{UP} \cap \mathsf{coUP}$ gives a *more intuitive* understanding of the latter complexity class: The solutions to the problems within that class can be expressed by a *unique* fixed point of some transformation that is carried out in polynomial time. Similar complexity classes to that one are FIXP and linear-FIXP = PPAD [109]. In the latter, *multiple* fixed points are allowed, and in FIXP the fixed points are even allowed to be *irrational*. The problem to find a Nash equilibrium in the two-party setting is known to be linear-FIXP-complete [109], and for three or more parties, it becomes FIXP-complete.

Since one can also factorize integers efficiently with quantum computers [225, 226], it is natural to ask how BQP relates to $\mathsf{P}_{\text{SRefCirc}}$. Unfortunately, little is known on that relation at the moment. Another question we can pose here is: What is the *quantum* analogue of the model presented here and what are the respective complexity classes? And finally, to come back to CTCs: Is it possible to embed *every* self-referential circuit in the CTC model described in Chapter 5?

Chapter 7

Application in other fields

We briefly show how the results from the previous chapters can be applied to different fields. We present non-causal correlations (see Chapter 4) in a model *without probabilities*. Probabilities, if not interpreted as *ignorance*, represent *counterfactualities*: events that did not occur, yet probabilities express what would have happen if "etwas geschehen wäre, was nicht geschehen ist"[1] [228].

Furthermore, we present a *local-realistic* model that explains quantum (and beyond) correlations by the help of closed time-like curves (see Chapter 5).

Finally, we extend the results from the Chapter 6 to the philosophy of language: *self-referential* sentences that are self-consistent.

7.1 Kolmogorov complexity

> In this matter of causality it is a great inconvenience that the real world is given to us once only. We cannot know what would have happened if something had been different. We cannot repeat an experiment changing just one variable; the hands of the clock will have moved, and the moons of Jupiter.
>
> John Stewart Bell [43]

A drawback of a formalism that incorporates probabilities beyond *ignorance* is that these probabilities are tied to events that did not happen. When performing an experiment, the result is not a probability distribution, but a series of results. A frequency analysis of the recorded outcomes can thereafter used to approximate the probability

[1] "something had happened that did not happen."

distribution over the different events. Clearly, this makes only sense if the *identical* experiment is repeated several times. But in what sense can one claim that an experiment is *identical* to another one? After all, the preceding experiments already produced outputs that we noted down in a lab log; so, a repeated experiment cannot be *identical*. Surely, one can *claim* that the lab log has no influence on the systems measured and on the measurement apparatuses. However, we can refrain from the use of such a claim by replacing the notion of probabilities with the one of the Kolmogorov [164, 169] complexity.

The Kolmogorov complexity $K(x)$ of some bit string x is defined as the *shortest* program for a fixed Turing machine that outputs x. Note that the running time is not considered here; this notion of complexity rather captures the structure of the bit string x. A clear drawback of that approach is the *fundamental impossibility* to compute the Kolmogorov complexity itself.

To be more accurate, let us fix some universal Turing machine $\mathcal{U}$, with respect to which we define the Kolmogorov complexity. For an *infinite* bit string

$$x = (x_0, x_1, \dots) \in \{0,1\}^\infty ,$$

we define $x_{[n]}$ as the bit string with identical bits on the first n positions, and with 0s on the rest:

$$x_{[n]} := (x_0, x_1, \dots, x_{n-1}, 0, 0, \dots) .$$

This allows us to study the *asymptotic behavior* in $n \to \infty$ of the Kolmogorov complexity of such a string. We define

$$K(x) \approx n :\Longleftrightarrow \lim_{n\to\infty} \left(\frac{K\left(a_{[n]}\right)}{n} \right) = 1 ,$$

and

$$K(x) \approx 1 :\Longleftrightarrow \lim_{n\to\infty} \left(\frac{K\left(a_{[n]}\right)}{n} \right) = 0 .$$

A bit string x that satisfies $K(x) \approx n$ is called *incompressible*. The intuition behind this naming is that for every additional bit of x we would like to compute, the program needs to be extended by a bit, asymptotically. In other words, the shortest program that outputs x simply recites x. Then again, a bit string x where $K(x) \approx 0$ holds is called *computable*: A program of *finite* length suffices to compute an arbitrary number of digits of x. An example of a computable string is the binary expansion of π, and an example of an incompressible string is Chaitin's [66] constant Ω which is also known as the *halting probability*. Incomputable bit strings are also called *algorithmically random* [66] — which can be understood as a notion of *randomness* without probabilities.

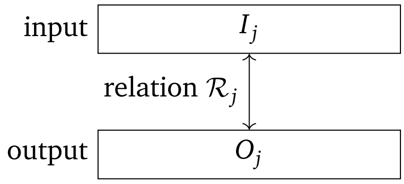


Figure 7.1. A party is modeled by a pair of bit strings and a relation that is satisfied by the blocks of these bit strings.

The *conditional* Kolmogorov complexity $K(x \mid a)$ is defined in a similar way, where the asymptotic behavior of $K(x_{[n]} \mid a_{[n]})$ for $n \to \infty$ is considered.

Interestingly, this Kolmogorov approach allows one to show similar statements as in the probability picture [79, 82]. If the inputs to the so-called PR box [205] are incomputable, then the outputs of that box must be incomputable as well [249–251]. Furthermore, the PR box leads to *complexity amplification* as we have shown elsewhere [35]. Here, we focus on the interplay between Kolmogorov complexity and the notion of causality and the resulting *non-causal* effects.

7.1.1 Operational definitions

A party is thought of as an experimenter performing some operation (see also Definition 6). We refrain from describing the experimenters choice or the outcomes by probability distributions, but instead stick at a description of the actual data only. Actual data are represented by bits strings and an infinite repetition of some experiment by an infinite bit string. In this section, we restrict ourselves to infinite repetitions (by that, the Kolmogorov complexity becomes *independent* of the choice of the universal Turing machine $\mathcal{U}$). This picture takes away all the *dynamics* and collapses to a *static* view: we are only interested in the complexity of the bit strings that satisfy some relations. A party S_j is modeled with two *infinite* bit strings, I^j (input) and O^j (output), where these bit strings are forced to satisfy the relation $\mathcal{R}_j$ on every successive block of finite length k_j, *i.e.*,

$$\forall q \geq 0 : \Big((I^j_{qk_j}, I^j_{qk_j+1}, \dots, I^j_{qk_j+k_j}), (O^j_{qk_j+1}, O^j_{qk_j+2}, \dots, O^j_{qk_j+k_j}) \Big) \in \mathcal{R}_j \, .$$

The relation $\mathcal{R}_j$ is called *local operation of* party S_j (see Figure 7.1).

We *define* I^j to be the *cause* of O^j.[2] Whenever we consider more than one party, we relate the bit strings among all parties with some global relation $\mathcal{E}$, that acts on finitely sized blocks as well.

Definition 29 (Consistency with respect to local operations)**.** A global relation $\mathcal{E}$ is called *consistent with respect to the local operations* if bit strings for all involved parties exist, where the bit strings mutually satisfy the local relations as well as $\mathcal{E}$.

[2] Note that this definition is arbitrary, but will prove itself helpful in later statements.

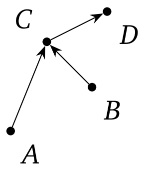


Figure 7.2. An example of a causal scenario with four parties: $(A,B) \preceq C \preceq D$. The arrows point from the algorithmic cause to the algorithmic effect.

For two parties S_j and S_k, we use the following definition of *causal past* and *causal future* (see Definition 6 for the counterfactual version thereof).

Definition 30 (Algorithmic causal past and algorihmic causal future). Party S_j (S_k) is said to be in the *algorithmic causal past (future)* of party S_k (S_j) if and only if

$$K(I^k \mid O^j) \not\approx K(I^k) \not\approx 0 \,.$$

This relation is expressed by $S_j \preceq S_k$.

Intuitively, the relation $S_j \preceq S_k$ holds if and only if the input of S_k is incomputable, yet less incomputable with the help of the output of S_j. This can be decomposed into the two statements:

- The input of S_k cannot be computed by S_k herself: $K(I^k) \not\approx 0$, and
- the input of S_k depends on the output of S_j: $K(I^k \mid O^j) \not\approx K(I^k)$.

Note that in the definition of causal past and causal future in Chapter 3, we postulated the distinction between free and non-free variables. Here, this distinction manifests itself in the distinction between *input* and *output* bit strings, and by the associated causal order between them.

7.1.2 Causal scenario

Using the model described above, we express a scenario where the relations between the parties are *causal*. A *causal scenario* is a scenario where the algorithmic causal relations between the parties reflect a *partial ordering* of the parties (see Figure 7.2).[3] A crucial implication of a causal scenario is that *at least one party is not in the algorithmic causal past of any other party*. If a scenario is not *causal*, then we call it *non-causal* (see Figure 7.3).

We briefly describe a bit-wise communication channel from S_1 to S_2. The resulting algorithmic causal relation will be $S_1 \preceq S_2$. Such a communication channel is described

[3] *Transitivity* follows from the assumption of a fixed causal relation between the input and output bit string of a party.

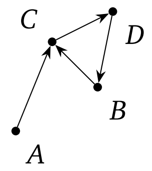


Figure 7.3. An example of a non-causal scenario.

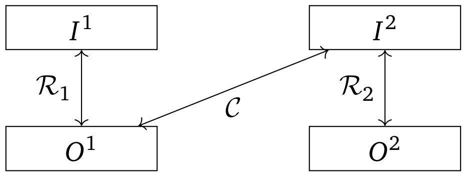


Figure 7.4. The global relation $\mathcal{C}$ ensures that the output of S_1 equals the input of S_2.

by the global relation $\mathcal{C}$:

$$\mathcal{C} = \{(0, x, x, y) \mid x, y \in \{0,1\}\}.$$

The whole setup (see Figure 7.4) is now described by the bit strings I^1, O^1, I^2, O^2 where

$$\forall q \geq 0 : \Big((I^1_{qk_1}, I^1_{qk_1+1}, \dots, I^1_{qk_1+k_1-1}), (O^1_{qk_1}, O^1_{qk_1+1}, \dots, O^1_{qk_1+k_1-1}) \Big) \in \mathcal{R}_1 ,$$
$$\forall q \geq 0 : \Big((I^1_{qk_2}, I^2_{qk_2+1}, \dots, I^2_{qk_2+k_2-1}), (O^2_{qk_2}, O^2_{qk_2+1}, \dots, O^2_{qk_2+k_2-1}) \Big) \in \mathcal{R}_2 ,$$
$$\forall q \geq 0 : \Big(I^1_q, O^1_q, I^2_q, O^2_q \Big) \in \mathcal{C} .$$

If we assume that both output bit strings are independent and incompressible, *i.e.*,

$$K(O^1, O^2) \approx 2n ,$$

then it follows that the input bit string of S_1 is the zero string $I^1 = (0, \dots)$, and $I^2 = O^1$. Since $K(I^2) \approx n$ and $K(I^2 \mid O^1) \approx 0$ hold, it follows from the definition of algorithmic causal past that $S_1 \preceq S_2$. Conversely, $K(I^1) \approx 0$ implies $S_2 \not\preceq S_1$. In words, the "sender" is in the algorithmic past of the receiver and *not vice versa*. The global relation $\mathcal{C}$ is clearly *consistent with respect to any choice of the local operations* of the parties (unless for the special case where $\mathcal{R}_1$ is a trivial relation that forces I^1 to be different from zero at every bit position).

7.1.3 Non-causal scenario

We discuss two non-causal scenarios. Assume the global relation $\mathcal{E}$ between two parties S_1, S_2 is

$$\mathcal{E} = \{(x, y, y, x) \mid x, y \in \{0,1\}\}.$$

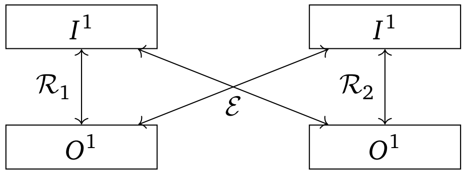


Figure 7.5. The global relation $\mathcal{E}$ represents a two-way channel that restricts the possible local relations $\mathcal{R}_1, \mathcal{R}_2$.

Such a global relation describes a *two-way channel* (see Figure 7.5). If we assume that the output bit strings of both parties are independent and incompressible, then we end up with the relation $S_1 \preceq S_2 \preceq S_1$. However, if we fix the local operations of both parties to be

$$\mathcal{R}_1 = \{(x,x) \mid x \in \{0,1\}\},$$
$$\mathcal{R}_2 = \{(x,x \oplus 1) \mid x \in \{0,1\}\},$$

i.e., the input and output bit strings are the same for S_1 and flipped for S_2, then no bit strings mutually satisfy all required conditions. Thus, $\mathcal{E}$ is *inconsistent* with respect to this choice of local operations — this particular case can be seen as an instantiation of the grandfather antinomy.

We now study a bit-wise global relation $\mathcal{E}'$ that leads to a non-causal scenario, yet that is *consistent* with respect to *any* bit-wise local operation. The latter is achieved by assuming that every party *independently* selects a bit-wise relation out of all four:

$$\mathcal{K}_{0,0} = \{(a,0) \mid a \in \{0,1\}\},$$
$$\mathcal{K}_{0,1} = \{(a,1) \mid a \in \{0,1\}\},$$
$$\mathcal{K}_{1,0} = \{(a,a) \mid a \in \{0,1\}\},$$
$$\mathcal{K}_{1,1} = \{(a,a \oplus 1) \mid a \in \{0,1\}\}.$$

If the relation $\mathcal{K}_{0,0}$ is applied, then the bit on the output bit string is a constant 0, *etc.* Assume three parties S_1, S_2, S_3 where every party holds an incompressible and independent bit string C^1, C^2, C^3, *i.e.*, $K(C^1, C^2, C^3) \approx 3n$. The idea is that these bit strings select the local relation to be applied on a given bit position, *i.e.* for party S_j

$$\forall q \geq 0 : \left(I^j_q, O^j_q\right) \in K_{C^j_{2q}, C^j_{2q+1}}.$$

An example is given in Figure 7.6. Since every pair of bits appears equally often in C^j (asymptotically speaking), and since the C^j strings are independent, all possible combinations of binary relations between the input and output bits of the parties arise. The global relation $\mathcal{E}$, discussed above, is clearly *inconsistent* with this set of local operations.

C^j | 10 00 01 11 10 0 $\cdots$

I^j | 0 1 1 0 1 $\cdots$

O^j | 0 0 1 1 1 $\cdots$

Figure 7.6. Example of a C^j and the related I^j, O^j from some party S_j.

However, the following bit-wise global relation is *consistent* with that choice of local operations:

$$\mathcal{E}' = \{((b \oplus 1)c, a, (c \oplus 1)a, b, (a \oplus 1)b, c) \mid a, b, c \in \{0,1\}\}\,.$$

This global relation is motivated by the example for non-causal correlations from Section 4.6.2. If we assume that all input bit strings are independent an incomputable, *i.e.*

$$K(I^1, I^2, I^3) \approx 3n\,,$$

then the only bit strings that satisfy all relations have the property

$$K(O^1) \approx K(O^2) \approx K(O^3) \not\approx 0\,,$$
$$K(O^1 \mid I^2, I^3) \approx K(O^2 \mid I^2, I^3) \approx K(O^3 \mid I^1, I^2) \approx 0\,.$$

Therefore, the algorithmic causal relations

$$(S_2, S_3) \preceq S_1\,,$$
$$(S_1, S_3) \preceq S_2\,,$$
$$(S_1, S_2) \preceq S_3\,,$$

are implied. These algorithmic causal relations are in contradiction with a party being in the algorithmic causal past of every other one, and by that, this scenario is *non-causal*. This global relation, furthermore, is consistent with *any* choice of binary local relations.

7.2 Bell non-local correlations from classical closed time-like curves

In the introduction (see Section 2.1), we extensively discussed local-realistic models and presented the quest to find such a model for quantum correlations. Then again, in Section 2.2 and in Chapter 5 we elaborated on the possibility of *closed time-like curves* and studied their dynamics. It is natural to ask whether *classical* (without making use of any quantum system) closed time-like curves allow for Bell non-local correlations. In other words, can Bell non-local correlations be simulated in a geometry of general relativity that contains closed time-like curves? This question is similar in nature, yet in our opinion much more realistic, to Lanczos' [165] speculation that the fundamental indistinguishability of elementary particles (lets say electrons) is a consequence of the existence of a single *ur*-electron, whilst all electrons are copies thereof that traveled through a closed time-like curve. Possibly one could understand *entangled* quantum systems in that way: The single systems are connected through closed time-like curves.

Here, we show that Deutsch's [93] formulation of closed time-like curves (CTCs) is a *local-realistic model* for quantum (and beyond) correlations. Deutsch initially presented his model to show that CTCs can be made consistent with quantum states (see Section 5.4.1). As for classical states, two possible paths have been explored. If, by classical states one understands *bits* as opposed to *random variables*, then, as shown by Deutsch [93], some choices of initial conditions lead to inconsistencies that cannot be overcome. If, on the other hand, one uses *random variables*, then the very same formalism as Deutsch presented can be applied (see, *e.g.*, Reference [6]).

7.2.1 Deutsch CTC model with random variables

Let CR be a classical system that is *causality respecting*, whose state is described by the probability distribution P_{CR}. Likewise, we denote by CV the *causality violating* classical system, which is in the state P_{CV}. The CV system travels on a CTC. Suppose now both systems, CR and CV, undergo some joint evolution $P_{\mathrm{CR}',\mathrm{CV}'|\mathrm{CR},\mathrm{CV}}$. The output systems thereafter are in the states $P_{\mathrm{CR}'}$ and $P_{\mathrm{CV}'}$, respectively (as calculated according to probability theory). In general, these states are correlated:

$$P_{\mathrm{CR}',\mathrm{CV}'} = \sum_{r,v} P_{\mathrm{CR}',\mathrm{CV}'|\mathrm{CR}=r,\mathrm{CV}=v} P_{\mathrm{CR}}(r) P_{\mathrm{CV}}(v) \,.$$

The Deutsch *consistency* condition is

$$P_{\mathrm{CV}} = \sum_{r'} P_{\mathrm{CR}'=r',\mathrm{CV}'} \,.$$

This condition ensures that the system traveling on the CTC does not "kill the grandfather." In other words, the distribution P_{CV} is a *fixed point* of the evolution with the input P_{CR}. Note that there might exist *more than one* fixed point. In that case, as Deutsch suggested, the problem of the information antinomy (see Section 3.3.1) is mitigated by *selecting* the fixed point that maximizes the entropy, which we denote by ρ_{CV}. The final state of the CR system ($\rho_{\mathrm{CR}'}$) is predicted by plugging in ρ_{CV} into the evolution (see Figure 7.7):

$$\rho_{\mathrm{CR}'} = \sum_{r,v,v'} P_{\mathrm{CR}',\mathrm{CV}'=v'|\mathrm{CR}=r,\mathrm{CV}=v} P_{\mathrm{CR}}(r) \rho_{\mathrm{CV}}(v) \,.$$

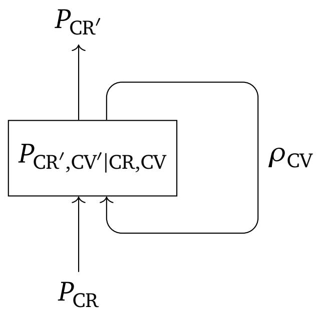


Figure 7.7. The output of the stochastic operation on the right side equals the input on the right side. The distribution ρ_{CV} is the distribution satisfies the just stated condition and that *maximizes* the entropy.

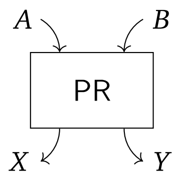


Figure 7.8. Alice and Bob, each, provide a bit to and receive a bit from the box, such that the parity of the outputs equals the product of the inputs.

7.2.2 PR box correlations from classical closed time-like curves

We present the setup with Deutsch CTCs based on random variables to reproduce PR box correlations. The PR box, named after Popescu and Rohrlich [205], is a system that idealizes Bell non-local correlations *beyond* the quantum regime. The box takes two binary inputs, A, B, from Alice and Bob, respectively, and produces two binary outputs, X, Y, to Alice and Bob, respectively (see Figure 7.8), such that the PR condition

$$AB = X \oplus Y$$

is satisfied.[4] Moreover, the PR box as a stochastic operation is defined as

$$P^{\mathrm{PR}}_{X,Y|A,B}(x, y, a, b) = \frac{\delta_{ab,x\oplus y}}{2} . \tag{7.1}$$

The behavior of this box cannot be simulated by Alice and Bob having access to *shared classical information* only, and where they do *not communicate*.

However, if Alice and Bob have additional access to a classical CTC whose dynamics is predicted by Deutsch's model, then they can simulate such a box by *local* interactions

[4]In the introduction, we already mentioned that expressions to describe a *Bell inequality* (see Section 2.1.3).

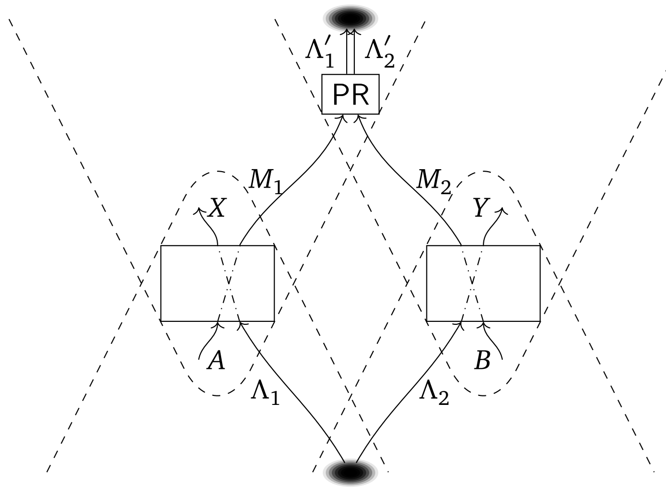


Figure 7.9. A PR box is simulated locally in the common future of Alice and Bob. The outputs of the PR box thereafter travel back in time trough a Deutschian closed time-like curve. The dashed lines represent the light cones. Time flows from bottom to top.

only. The latter means that both parties do *not communicate*, unless they are in the vicinity of each other. We quickly present the model, and thereafter discuss some features thereof.

Assume that the laboratories of Alice and Bob are space-like separated (they cannot talk to each other). Alice chooses an input A she wishes to input to the PR box, and so does Bob: B. Both inputs are random variables. Furthermore, every party receives some random variable (Λ_1 and Λ_2, respectively) from their common past. Now, Alice simply uses $X = \Lambda_1$ as her output and Bob uses $Y = \Lambda_2$ as his output, and, as a surprise, the PR box condition $AB = X \oplus Y$ holds (as we are going to see). The inputs A, B of both parties are sent to the *common* future, where they undergo some interaction. Once the variables arrived in the common future, any interaction between them is possible; for what should prevent any *local* interaction? We choose the *local* interaction in the *common future* of Alice and Bob to be such that it simulates a PR box. This means, after the *local* interactions, the random variables are in a state as predicted by the PR box with inputs A, B. The pair of random variables thereafter enters a mouth of a wormhole to travel back in time, where it emerges in the common past of both parties as Λ_1 and Λ_2. There, the random variables are split and each party receives one of them. This setup is depicted in Figure 7.9.

Mathematically, this setup is the following. Alice, and Bob, each perform an operation that swaps both of their random variables:

$$P^{\mathrm{SWAP}}_{Q',R'|Q,R}(q',r',q,r) = \delta_{q',r}\delta_{r',q}\,.$$

Thereafter, the PR box (see Equation (7.1)) is applied to the systems that later enter the CTC. Thus, the total dynamics can be expressed by

$$\begin{aligned}P_{X,Y,\Lambda_1',\Lambda_2'|A,B,\Lambda_1,\Lambda_2}(x,y,\lambda_1',\lambda_2',a,b,\lambda_1,\lambda_2) &= \sum_{\mu_1,\mu_2} P^{\mathrm{PR}}_{\Lambda_1',\Lambda_2'|M_1,M_2}(\lambda_1',\lambda_2',\mu_1,\mu_2)\times \\ &\quad P^{\mathrm{SWAP}}_{X,M_1|A,\Lambda_1}(x,\mu_1,a,\lambda_1)P^{\mathrm{SWAP}}_{Y,M_2|B,\Lambda_2}(y,\mu_2,b,\lambda_2) \\ &= \sum_{\mu_1,\mu_2} \frac{\delta_{\mu_1\mu_2,\lambda_1'\oplus\lambda_2'}}{2}\delta_{x,\lambda_1}\delta_{\mu_1,a}\delta_{y,\lambda_2}\delta_{\mu_2,b} \\ &= \frac{\delta_{ab,\lambda_1'\oplus\lambda_2'}}{2}\delta_{x,\lambda_1}\delta_{y,\lambda_2}\,.\end{aligned}$$

For a given input of the parties P_A, P_B, the consistency condition is

$$\begin{aligned}\rho_{\Lambda_1',\Lambda_2'}(\lambda_1',\lambda_2') &= \sum_{a,b,x,y,\lambda_1,\lambda_2} \frac{\delta_{ab,\lambda_1'\oplus\lambda_2'}}{2}\delta_{x,\lambda_1}\delta_{y,\lambda_2}P_A(a)P_B(b)\rho_{\Lambda_1',\Lambda_2'}(\lambda_1,\lambda_2) \\ &= \sum_{a,b}\frac{\delta_{ab,\lambda_1'\oplus\lambda_2'}}{2}P_A(a)P_B(b)\,,\end{aligned}$$

which yields a *unique* solution.

For deterministic input distributions

$$\begin{aligned}P_A'(a) &= \delta_{a,\bar{a}}\,,\\ P_B'(b) &= \delta_{b,\bar{b}}\,,\end{aligned}$$

the solution becomes

$$\rho_{\Lambda_1',\Lambda_2'}(\lambda_1',\lambda_2') = \frac{\delta_{\bar{a}\bar{b},\lambda_1'\oplus\lambda_2'}}{2}\,.$$

Thus, for an deterministic input, the solution $\rho_{\Lambda',\Lambda'}$ simply carries the output of a PR box when queried with these inputs. When this solution is plugged in to calculate the distribution over X and Y, one finds that $P_{X,Y} = \rho_{\Lambda_1',\Lambda_2'}$. Thus, the output from the PR box is simply forwarded to the parties, and the reduced probability distribution $P_{X,Y|A,B}$, when queried with deterministic inputs, yields the PR box distribution.

That this is the case can also be seen by noting that the swap operations performed by the parties renders the CTC to an *open* time-like curve [202]: The systems that travel on the CTC do *not self-interact*.

At this point, one might wonder what happens if the parties perform *other* operations than the swap. One can show that for *any* operation they choose to perform, the resulting distribution is *always non-signaling*. This means that by diverting from the above protocol, each party will never be able to send signals at a velocity faster than light to the other — in accordance with the theory of relativity.

Moreover, what can ask what happens if the implemented operation in the common future of the parties does not simulate a PR box, but is a swap operation, for instance.

In that case, clearly, the setup would become signaling: A travels to the common future of Alice and Bob, there it swaps position with B, falls in the wormhole, and emerges in the common past of both parties to travel to Bob. One might mitigate this issue of signaling by enforcing that Λ_2 should contain no information about A, and *vice versa*, *i.e.*, by a non-signalling assumption.

Since in the simulation protocol described above, one output of the PR box does not contain any information about any input, the just proposed *non-signaling* assumption still allows the parties to generate PR box correlations in a *local* way.

The above protocol to generate PR box correlations resembles very much the approach of *parallel lives* [54] (see Section 2.1.5). There, the "correct" parties meet such that the correlations are obtained. Here, instead of sending physical systems, the parties themselves could meet in the common past and fall into the wormhole. By this, the reason for having PR correlations *at this moment* is that the parties will *meet in the future*. So, the correlations are "borrowed" from the future and observed in the past.

7.3 Causal loops in language

> All Cretans are liars.
> attributed to Epimenides from Crete [218]

In Chapter 6, we constructed a model of computation where all values are neither under- nor overdetermined. These assumptions reflect the necessity of avoiding the *grandfather* as well as the *information antinomy*. We saw that, in the case of the computational model, self-referential expressions are possible under these requirements.

The same approach can be pursued in language. The famous *liar paradox*[5] reads

"I am lying.",

which, from a logical point of view and for the purpose of this section, is equivalent to

P : "This sentence is false.".

If the proposition P were to be *true*, then P states that P must be false, which then again would imply P to be true, *etc.* So, the *liar paradox* introduces a *contradiction* in its own statement — which is essentially the *grandfather antinomy*.

Once this analogy to the grandfather antinomy is identified, we can ask for a sentence analogous to the *information antinomy*:

Q : "This sentence is true.".

[5] It is actually an *antinomy*.

If the sentence Q were to be true, then it would say of itself, that it is true — no contradiction. Then again, if Q were to be false, then it says of itself that it is false — again no contradiction. However, for what is crucial in the information antinomy, all information at hand *do not specify* the truth value of Q, making it useless for most other purposes than for using in this and the previous two sentences of this thesis.

But is it possible to come up with a sentence, or a list of sentences, that are *self-referential*, yet where the grandfather and information antinomies do not arise? We propose the following two sentences:

R_1 :"The truth values of this and the following sentence are different."

R_2 :"The truth values of this and the following sentence are the same."

This system of sentence has now a *unique* assignment of truth values to R_1 and R_2.

At this point let us briefly present a similar system of sentences by the French philosopher Jean Buridan [62], to prove the existence of "god:"[6]

B_1 :"The flying spaghetti monster exists."

B_2 :"Neither of the two sentences is true."

Assume B_2 to be true. From this follows that B_2 is false, which is the grandfather antinomy. The other possibility is to assume B_2 to be false. By that, at least one sentence must be true: *The flying spaghetti monster exists*. The grandfather nor the information antinomy arises in this system of sentences.

Note that this bypasses Tarski's [231] approach to *truth* in language, where a *meta* language is needed in order to make non-contradictory self-referential statements. That Tarski's approach is in some cases too restrictive, was also demonstrated by [140] (see also Reference [141]), who construct similar systems of sentences.

One might wonder whether more interesting *self-referential* statements were possible — not necessarily in language, but for instance in set theory. Would it be possible to define sets by referring to themselves, such that the sets are well-defined? If yes, would these sets be consistent with *ZFC set theory*? We leave this question open.

[6]We slightly alter the example to "prove" the existence of the Pastafarian's god.

Chapter 8

Conclusions and open question

We explored some parts of the *logically consistent* world *beyond the causal*. By *beyond the causal* we mean that the assumption of causal structures being *acyclic* and *definite* is dropped. As a replacement, we used the assumption of *logical consistency*: Probabilities or dynamics are not contradictory and well-defined. A necessity to perform studies in this field is a definition of *causal relations*; this was done in the chapter *On causality*. A first result of this expedition is: The classical and logically consistent world beyond the causal is *non-empty*. This result can be understood as the spark for this thesis: How does this world look like? We explored this *classical* world from three points of view: *correlations*, *time travel*, and *computation*. By having local assumptions of classical nature only, we described a framework for correlations without causal order. The resources of this framework were then characterized in two ways: by polytopes and by fixed-point theorems. The latter characterization gave some insight on *logical consistency* and allowed us to describe *reversible* dynamics.

From that, we went over to the realm of general relativity and used the tools at hand to study closed time-like curves. Whilst geometries with closed time-like curves are consistent with general relativity, noting (not to say *little*) with respect to their physical nature and the associated dynamics is known. We were able to extend previous results on closed time-like curves — these results show the *innocence* of closed time-like curves with respect to logical problems. We started these studies at the assumptions that any dynamics must be *consistent*, and that physics in local regions remain *invariant* under the existence or absence of closed time-like curves. From these two assumptions we were able to show that the *Cauchy* problem is always well-posed: *unambiguous* dynamics are predicted from the boundary conditions. Note that this was not the case in previous considerations of closed time-like curves (see, *e.g.*, Reference [122]). A second result strengthens the compatibility of closed time-like curves with general relativity: All dynamics can be made *reversible*. This at hand, we *explicitly* described some closed time-like curves dynamics between three local space-time regions, where every region is in the *future* and in the *past* of every other region. We then compared our model with previously known models of closed time-like curves, and showed that while the other models are problematic in some ways (non-linear, not reversible, ad-hoc solutions to the Cauchy problem), ours is not.

From this, we continued to study the computational power of the framework for the non-causal correlations and of the closed time-like curves. For that purpose, we designed a circuit model of computation that allows for causal loop. The model of computation is shown to lead to advantages in query complexity and in computational complexity, when compared to the traditional circuit model. We identified the complexity class of problems efficiently solvable by that model, and it turned out to be $\mathsf{UP} \cap \mathsf{coUP} \subseteq \mathsf{NP}$ — a class which is believed to be not too powerful, yet it contains the problem of *integer factorization*. This class is believed to be strictly contained in NP and strictly contain P. The results found here, can be used to strengthen the statement that closed time-like curves are *unproblematic*: They do not violate the NP-hardness assumption which conjectures that no *physically realizable model of computation* can solve NP-complete problems. The previously designed models for closed time-like curves, however, are *drastically* more powerful (PSPACE, and $\mathsf{BPP}_{\mathrm{path}}$), and violate this assumption by far. At the end of the chapter on this model, we explicitly designed a circuit to factorize integers efficiently within that model.

We continued with a short chapter on the applications of these results to different fields. One of the main messages of this thesis is that self-referentiality is *tame* and in some cases it can even be *helpful*, but not too helpful. So, in that chapter we showed the compatibility of our results with the notion of Kolmogorov complexity — the *same* statements can be made; we showed how some closed time-like curves could be used in order to simulate *Bell non-local correlations* with classical information and local interactions only; and finally, we briefly mentioned self-referentiality in language, *e.g.*, the liar paradox.

8.1 Open questions

Be briefly state some open question that we group is two classes: *computer science* and the *foundations of physics*. One of the main questions in the *foundation of physics* is to ask: Which processes are in principle physically realizable? Currently it is unclear whether the dynamics described in this thesis are realizable or whether they are just an artefact of the theories. Yet, the study of these dynamics, even if they were not to exist, have some other implications and open the question: Can we simulate quantum theory within general relativity with a *circular* space-time geometry? We gave some partial answer to this second question in the last chapter. Towards a better understating of the interplay between quantum and relativity theory, a better understanding of the concept of “time” seems necessary (see, *e.g.*, References [108, 126, 143, 195, 209, 254]).

A questions in *computer science* to look at is: What information-processing tasks become possible or more efficient with the models studied here? For instance, it is not known whether any of the models presented does allow for some advantage in *cryptographic* setups. Another question is to study *quantum closed time-like curves* and the computational power thereof. Finally, what are the relations of $\mathsf{UP} \cap \mathsf{coUP}$ to P, BQP, and to NP? Our results give an interpretation of $\mathsf{UP} \cap \mathsf{coUP}$ in terms of fixed points, which might be a new path towards relating this class to others.

Bibliography

[1] Scott Aaronson. Guest column: NP-complete problems and physical reality. *ACM SIGACT News*, 36(1):30, March 2005. doi:10.1145/1052796.1052804.

[2] Scott Aaronson. Quantum computing, postselection, and probabilistic polynomial-time. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 461(2063):3473–3482, November 2005. doi:10.1098/rspa.2005.1546.

[3] Scott Aaronson. Time: Different from space. `http://www.scottaaronson.com/blog/?p=368`, December 2008.

[4] Scott Aaronson. My 116-page survey article on P vs. NP: better late than never. `http://www.scottaaronson.com/blog/?p=3095`, January 2017.

[5] Scott Aaronson. Complexity Zoo. `https://complexityzoo.uwaterloo.ca/Complexity_Zoo`, 2017. Accessed: 2017-01-18.

[6] Scott Aaronson and John Watrous. Closed timelike curves make quantum and classical computing equivalent. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 465(2102):631–647, February 2009. doi:10.1098/rspa.2008.0350.

[7] Scott Aaronson, Mohammad Bavarian, and Giulio Gueltrini. Computability theory of closed timelike curves. *preprint arXiv:1609.05507 [quant-ph]*, September 2016. URL `http://arxiv.org/abs/1609.05507`.

[8] Alastair A Abbott, Christina Giarmatzi, Fabio Costa, and Cyril Branciard. Multipartite causal correlations: polytopes and inequalities. *Physical Review A*, 94(3):032131, September 2016. doi:10.1103/PhysRevA.94.032131.

[9] Daniel S Abrams and Seth Lloyd. Nonlinear quantum mechanics implies polynomial-time solution for NP-complete and #P problems. *Physical Review Letters*, 81(18):3992–3995, November 1998. doi:10.1103/PhysRevLett.81.3992.

[10] Manindra Agrawal, Neeraj Kayal, and Nitin Saxena. PRIMES is in P. *Annals of Mathematics*, 160(2):781–793, September 2004. doi:10.4007/annals.2004.160.781.

[11] Doyeol Ahn, Casey R Myers, Timothy C Ralph, and Robert B Mann. Quantum-state cloning in the presence of a closed timelike curve. *Physical Review A*, 88 (2):022332, August 2013. doi:10.1103/PhysRevA.88.022332.

[12] John-Mark A Allen, Jonathan Barrett, Dominic C Horsman, Ciaran M Lee, and Robert W Spekkens. Quantum common causes and quantum causal models. *preprint arXiv:1609.09487 [quant-ph]*, September 2016. URL `http://arxiv.org/abs/1609.09487`.

[13] Mateus Araújo. Understanding Bell's theorem part 2: the nonlocal version. `http://mateusaraujo.info/2016/07/21/understanding-bells-theorem-part-2-the-nonlocal-version/`, 2016.

[14] Mateus Araújo and Adrien Feix. personal communication, 2014.

[15] Mateus Araújo, Fabio Costa, and Časlav Brukner. Computational advantage from quantum-controlled ordering of gates. *Physical Review Letters*, 113(25):250402, December 2014. doi:10.1103/PhysRevLett.113.250402.

[16] Mateus Araújo, Cyril Branciard, Fabio Costa, Adrien Feix, Christina Giarmatzi, and Časlav Brukner. Witnessing causal nonseparability. *New Journal of Physics*, 17(10):102001, October 2015. doi:10.1088/1367-2630/17/10/102001.

[17] Mateus Araújo, Adrien Feix, Miguel Navascués, and Časlav Brukner. A purification postulate for quantum mechanics with indefinite causal order. *preprint arXiv:1611.08535 [quant-ph]*, November 2016. URL `http://arxiv.org/abs/1611.08535`.

[18] Sanjeev Arora and Boaz Barak. *Computational Complexity. A Modern Approach*. Cambridge University Press, New York, 2009. ISBN 9780521424264.

[19] Alain Aspect, Philippe Grangier, and Gérard Roger. Experimental tests of realistic local theories via Bell's theorem. *Physical Review Letters*, 47(7):460–463, August 1981. doi:10.1103/PhysRevLett.47.460.

[20] Alain Aspect, Jean Dalibard, and Gérard Roger. Experimental test of Bell's inequalities using time- varying analyzers. *Physical Review Letters*, 49(25):1804–1807, December 1982. doi:10.1103/PhysRevLett.49.1804.

[21] David Avis, David Bremner, and Raimund Seidel. How good are convex hull algorithms? *Computational Geometry*, 7(5-6):265–301, April 1997. doi:10.1016/S0925-7721(96)00023-5.

[22] Jean-Daniel Bancal, Stefano Pironio, Antonio Acín, Yeong-Cherng Liang, Valerio Scarani, and Nicolas Gisin. Quantum non-locality based on finite-speed causal influences leads to superluminal signalling. *Nature Physics*, 8(12):867–870, October 2012. doi:10.1038/nphys2460.

[23] C Bradford Barber, David P Dobkin, and Hannu Huhdanpaa. The quickhull algorithm for convex hulls. *ACM Transactions on Mathematical Software*, 22(4): 469–483, December 1996. doi:10.1145/235815.235821.

[24] Tomer Jack Barnea, Jean-Daniel Bancal, Yeong-Cherng Liang, and Nicolas Gisin. Tripartite quantum state violating the hidden-influence constraints. *Physical Review A*, 88(2):022123, August 2013. doi:10.1103/PhysRevA.88.022123.

[25] Peter Baumann. *Erkenntnistheorie*. J B Metzler, Stuttgart, 2006. ISBN 978-3-476-02134-2.

[26] Veronika Baumann. *Semiquantum Correlations and Indefinite Causal Structures for Two Laboratories*. Master's thesis, Universität Wien, August 2014. URL `http://othes.univie.ac.at/34190/`.

[27] Veronika Baumann and Časlav Brukner. Appearance of causality in process matrices when performing fixed-basis measurements for two parties. *Physical Review A*, 93(6):062324, June 2016. doi:10.1103/PhysRevA.93.062324.

[28] Ämin Baumeler and Stefan Wolf. Perfect signaling among three parties violating predefined causal order. In *2014 IEEE International Symposium on Information Theory*, pages 526–530, Piscataway, June 2014. IEEE. ISBN 978-1-4799-5186-4. doi:10.1109/ISIT.2014.6874888.

[29] Ämin Baumeler and Stefan Wolf. Causality — Complexity — Consistency: Can space-time be based on logic and computation? *preprint arXiv:1602.06987 [quant-ph]. To appear in* Proceedings of Workshop in Time in Physics*, ETH Zürich.*, February 2016. URL `http://arxiv.org/abs/1602.06987`.

[30] Ämin Baumeler and Stefan Wolf. Non-causal computation. *preprint arXiv:1601.06522 [quant-ph]*, January 2016. URL `http://arxiv.org/abs/1601.06522`.

[31] Ämin Baumeler and Stefan Wolf. Device-independent test of causal order and relations to fixed-points. *New Journal of Physics*, 18(3):035014, April 2016. doi:10.1088/1367-2630/18/3/035014.

[32] Ämin Baumeler and Stefan Wolf. The space of logically consistent classical processes without causal order. *New Journal of Physics*, 18(1):013036, January 2016. doi:10.1088/1367-2630/18/1/013036.

[33] Ämin Baumeler and Stefan Wolf. Computational tameness of classical non-causal models. *preprint arXiv:1611.05641 [quant-ph]*, November 2016. URL http://arxiv.org/abs/1611.05641.

[34] Ämin Baumeler, Adrien Feix, and Stefan Wolf. Maximal incompatibility of locally classical behavior and global causal order in multiparty scenarios. *Physical Review A*, 90(4):042106, October 2014. doi:10.1103/PhysRevA.90.042106.

[35] Ämin Baumeler, Charles Alexandre Bédard, Gilles Brassard, and Stefan Wolf. Kolmogorov amplification from Bell correlations. *unpublished*, 2017.

[36] Ämin Baumeler, Fabio Costa, Timothy C Ralph, Stefan Wolf, and Magdalena Zych. Reversible time travel with freedom of choice. *preprint arXiv:1703.00779 [gr-qc]*, March 2017. URL http://arxiv.org/abs/1703.00779.

[37] Ämin Baumeler, Fabio Costa, Timothy C Ralph, Stefan Wolf, and Magdalena Zych. Linear and reversible closed time-like curves. *unpublished*, 2017.

[38] Ämin Baumeler, Julien Degorre, and Stefan Wolf. Simulating PR-box correlations with classical closed time-like curves. *unpublished*, 2017.

[39] Helen Beebee, Christopher Hitchcock, and Peter Menzies, editors. *The Oxford Handbook of Causation*. Oxford University Press, Oxford, 2012. ISBN 978-0-19-964258-8.

[40] John Stewart Bell. On the Einstein-Podolsky-Rosen paradox. *Physics*, 1(3):195–200, 1964.

[41] John Stewart Bell. On the problem of hidden variables in quantum mechanics. *Reviews of Modern Physics*, 38(3):447–452, July 1966. doi:10.1103/RevModPhys.38.447.

[42] John Stewart Bell. The theory of local beables. Technical report, CERN, July 1975. Presented at the sixth GIFT Seminar, Jaca, 2–7 June 1975, and reproduced in *Epistemological Letters*, March 1976, and in *dialectica*, June 1985.

[43] John Stewart Bell. Free variables and local causality. *Epistemmological Letters*, February 1977. doi:10.1111/j.1746-8361.1985.tb01249.x. Reproduced in *dialectica*, June 1985.

[44] John Stewart Bell. *Speakable and Unspeakable in Quantum Mechanics*. Cambridge University Press, Cambridge, 2 edition, 2004. ISBN 978-0-521-52338-7.

[45] Charles H Bennett. The thermodynamics of computation—a review. *International Journal of Theoretical Physics*, 21(12):905–940, December 1982. doi:10.1007/BF02084158.

[46] Charles H Bennett and Benjamin Schumacher. Simulated time travel, teleportation without communication, and how to conduct a romance with someone who has fallen into a black hole, May 2005. URL http://web.archive.org/web/20070401092204/http://www.research.ibm.com/people/b/bennetc/QUPONBshort.pdf. Talk at QUPON, 2005, Vienna, Austria.

[47] Charles H Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K Wootters. Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. *Physical Review Letters*, 70(13):1895–1899, March 1993. doi:10.1103/PhysRevLett.70.1895.

[48] Charles H Bennett, Debbie Leung, Graeme Smith, and John A Smolin. Can closed timelike curves or nonlinear quantum mechanics improve quantum state discrimination or help solve hard problems? *Physical Review Letters*, 103(17): 170502, October 2009. doi:10.1103/PhysRevLett.103.170502.

[49] David Bohm. Quantum theory of the measurement process. In *Quantum Theory*, chapter 22, pages 583–623. Dover Publications, Inc., New York, 1989. ISBN 0-486-65969-0.

[50] David Bohm and Yakir Aharonov. Discussion of experimental proof for the paradox of Einstein, Rosen, and Podolsky. *Physical Review*, 108(4):1070–1076, November 1957. doi:10.1103/PhysRev.108.1070.

[51] Niels Bohr. Can quantum-mechanical description of physical reality be considered complete? *Physical Review*, 48(8):696–702, October 1935. doi:10.1103/PhysRev.48.696.

[52] Cyril Branciard. Witnesses of causal nonseparability: an introduction and a few case studies. *Scientific Reports*, 6:26018, May 2016. doi:10.1038/srep26018.

[53] Cyril Branciard, Mateus Araújo, Adrien Feix, Fabio Costa, and Časlav Brukner. The simplest causal inequalities and their violation. *New Journal of Physics*, 18 (1):013008, December 2015. doi:10.1088/1367-2630/18/1/013008.

[54] Gilles Brassard and Paul Raymond-Robichaud. Can free will emerge from determinism in quantum theory? In A Suarez and P Adams, editors, *Is Science Compatible with Free Will?*, chapter 4, pages 41–61. Springer New York, New York, NY, 2013. ISBN 978-1-4614-5212-6. doi:10.1007/978-1-4614-5212-6_4.

[55] Gilles Brassard, Richard Cleve, and Alain Tapp. Cost of exactly simulating quantum entanglement with classical communication. *Physical Review Letters*, 83(9): 1874–1877, August 1999. doi:10.1103/PhysRevLett.83.1874.

[56] Harvey R Brown and Christopher G Timpson. Bell on Bell's theorem: the changing face of nonlocality. *preprint arXiv:1501.03521 [quant-ph]*, December 2014. URL http://arxiv.org/abs/1501.03521.

[57] Časlav Brukner. Bounding quantum correlations with indefinite causal order. *New Journal of Physics*, 17(8):083034, August 2015. doi:10.1088/1367-2630/17/8/083034.

[58] Časlav Brukner, Marek Zukowski, and Anton Zeilinger. The essence of entanglement. *preprint arXiv:0106119 [quant-ph]*, June 2001. URL http://arxiv.org/abs/quant-ph/0106119.

[59] Todd A Brun and Mark M Wilde. Perfect state distinguishability and computational speedups with postselected closed timelike curves. *Foundations of Physics*, 42(3):341–361, March 2012. doi:10.1007/s10701-011-9601-0.

[60] Todd A Brun, Jim Harrington, and Mark M Wilde. Localized closed timelike curves can perfectly distinguish quantum states. *Physical Review Letters*, 102 (21):210402, May 2009. doi:10.1103/PhysRevLett.102.210402.

[61] Todd A Brun, Mark M Wilde, and Andreas Winter. Quantum state cloning using Deutschian closed timelike curves. *Physical Review Letters*, 111(19):190401, November 2013. doi:10.1103/PhysRevLett.111.190401.

[62] John Buridan. *John Buridan On Self-Reference: Chapter 8 Eight of Buridan's 'Sophismata,' with a Translation, an Introduction, and a Philosophical Commentary*. Cambridge University Press, Cambridge, 1982. ISBN 978-0-521-28864-4.

[63] Adán Cabello. Interpretations of quantum theory: a map of madness. *preprint arXiv:1509.04711 [quant-ph]*, page 3, September 2015. URL http://arxiv.org/abs/1509.04711.

[64] Esteban Castro-Ruiz, Flaminia Giacomini, and Časlav Brukner. Entanglement of quantum clocks through gravity. *preprint arXiv:1507.01955 [quant-ph]*, July 2015. URL http://arxiv.org/abs/1507.01955.

[65] Nicolas J Cerf, Nicolas Gisin, Serge Massar, and Sandu Popescu. Simulating maximal quantum entanglement without communication. *Physical Review Letters*, 94 (22):220403, jun 2005. doi:10.1103/PhysRevLett.94.220403.

[66] Gregory J Chaitin. A theory of program size formally identical to information theory. *Journal of the ACM*, 22(3):329–340, July 1975. doi:10.1145/321892.321894.

[67] NV. Chernikova. Algorithm for finding a general formula for the non-negative solutions of a system of linear inequalities. *USSR Computational Mathematics*

and Mathematical Physics, 5(2):228–233, January 1965. doi:10.1016/0041-5553(65)90045-5.

[68] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Quantum circuit architecture. *Physical Review Letters*, 101(6):060401, August 2008. doi:10.1103/PhysRevLett.101.060401.

[69] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Transforming quantum operations: quantum supermaps. *EPL (Europhysics Letters)*, 83(3): 30004, August 2008. doi:10.1209/0295-5075/83/30004.

[70] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Theoretical framework for quantum networks. *Physical Review A*, 80(2):022339, August 2009. doi:10.1103/PhysRevA.80.022339.

[71] Giulio Chiribella, Giacomo Mauro D'Ariano, Paolo Perinotti, and Benoit Valiron. Quantum computations without definite causal structure. *Physical Review A*, 88 (2):022318, August 2013. doi:10.1103/PhysRevA.88.022318.

[72] Man-Duen Choi. Completely positive linear maps on complex matrices. *Linear Algebra and its Applications*, 10(3):285–290, June 1975. doi:10.1016/0024-3795(75)90075-0.

[73] Boris Semyonovich Cirel'son. Quantum generalizations of Bell's inequality. *Letters in Mathematical Physics*, 4(2):93–100, March 1980. doi:10.1007/BF00417500.

[74] Samuel Clarke. Mr. Leibnitz's fifth paper, being an answer to Dr. Clarke's fourth reply. In *A Collection of Papers, Which Passed Between the Late Learned Mr. Leibnitz, and Dr. Clare, in the Years 1715 and 1716*, pages 155–279. James Knapton, London, 1717.

[75] Samuel Clarke. Cinquiéme écrit de Mr. Leibnitz, ou réponse au quatriéme écrit Anglois. In *A Collection of Papers, Which Passed Between the Late Learned Mr. Leibnitz, and Dr. Clare, in the Years 1715 and 1716*, pages 154–278. James Knapton, London, 1717.

[76] Samuel Clarke. *A Collection of Papers, Which Passed Between the Late Learned Mr. Leibnitz, and Dr. Clare, in the Years 1715 and 1716*. James Knapton, London, 1717.

[77] John F Clauser, Michael A Horne, Abner Shimony, and Richard A Holt. Proposed experiment to test local hidden-variable theories. *Physical Review Letters*, 23(15): 880–884, October 1969. doi:10.1103/PhysRevLett.23.880.

[78] Roger Colbeck and Renato Renner. No extension of quantum theory can have improved predictive power. *Nature Communications*, 2:411, August 2011. doi:10.1038/ncomms1416.

[79] Roger Colbeck and Renato Renner. Free randomness can be amplified. *Nature Physics*, 8(6):450–454, May 2012. doi:10.1038/nphys2300.

[80] Roger Colbeck and Renato Renner. A short note on the concept of free choice. *preprint arXiv:1302.4446 [quant-ph]*, February 2013. URL `http://arxiv.org/abs/1302.4446`.

[81] Timoteo Colnaghi, Giacomo Mauro D'Ariano, Stefano Facchini, and Paolo Perinotti. Quantum computation with programmable connections between gates. *Physics Letters A*, 376(45):2940–2943, October 2012. doi:10.1016/j.physleta.2012.08.028.

[82] John Conway and Simon Kochen. The free will theorem. *Foundations of Physics*, 36(10):1441–1473, November 2006. doi:10.1007/s10701-006-9068-6.

[83] John Conway and Simon Kochen. The strong free will theorem. *Notices of the AMS*, 56(2):226–232, 2009.

[84] Sandro Coretti, Esther Hänggi, and Stefan Wolf. Nonlocality is transitive. *Physical Review Letters*, 107(10):100402, August 2011. doi:10.1103/PhysRevLett.107.100402.

[85] Fabio Costa. *Local and Causal Structures in Quantum Theory*. PhD thesis, Universtiät Wien, 2013. URL `http://othes.univie.ac.at/29237/`.

[86] Allan H Coxon. *The Fragments of Parmenides: Revised and Expanded Edition*. Parmenides Publishing, Las Vegas, 2009. ISBN 978-1-930972-67-4.

[87] O Costa de Beauregard. Le "paradoxe" des corrélations d'Einstein et de Schrödinger et l'épaisseur temporelle de la transistion quantique. *Dialectica*, 19(3-4):280–289, December 1965. doi:10.1111/j.1746-8361.1965.tb00474.x.

[88] O Costa de Beauregard. Time symmetry and interpretation of quantum mechanics. *Foundations of Physics*, 6(5):539–559, October 1976. doi:10.1007/BF00715107.

[89] O Costa de Beauregard. Time symmetry and the Einstein paradox. *Il Nuovo Cimento B*, 42(1):41–64, November 1977. doi:10.1007/BF02906749.

[90] Baruch de Spinoza. Ethica. In *Opera Posthuma*, chapter 1. Jan Rieuwertsz, Amsterdam, 1677.

[91] Julien Degorre, Sophie Laplante, and Jérémie Roland. Simulating quantum correlations as a distributed sampling problem. *Physical Review A*, 72(6):062314, December 2005. doi:10.1103/PhysRevA.72.062314.

[92] Wolfgang Detel. *Erkenntnis- und Wissenschaftstheorie*. Reclam, Stuttgart, 2007. ISBN 978-3-15-018471-4.

[93] David Deutsch. Quantum mechanics near closed timelike lines. *Physical Review D*, 44(10):3197–3217, November 1991. doi:10.1103/PhysRevD.44.3197.

[94] David Deutsch. *The Fabric of Reality: The Science of Parallel Universes and Its Implications*. Viking Adult, New York, 1997. ISBN 978-0-713-99061-4.

[95] Bryce S DeWitt. Quantum mechanics and reality. *Physics Today*, 23(9):30–35, September 1970. doi:10.1063/1.3022331.

[96] Michael Dummett. *Truth and Other Enigmas*. Harvard University Press, Cambridge, 1978. ISBN 978-0-67-491076-8.

[97] Fernando Echeverria, Gunnar Klinkhammer, and Kip S Thorne. Billiard balls in wormhole spacetimes with closed timelike curves: classical theory. *Physical Review D*, 44(4):1077–1099, August 1991. doi:10.1103/PhysRevD.44.1077.

[98] Albert Einstein. Über das Relativitätsprinzip und die aus demselben gezogene Folgerungen. *Jahrbuch der Radioaktivität und Elektronik*, 4:411–462, 1907.

[99] Albert Einstein. Die formale Grundlage der allgemeinen Relativitätstheorie. In Georg Reimer, editor, *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, volume Zweiter Halbband, pages 1030–1085. Verlag der Königlichen Akademie der Wissenschaften, Berlin, 1914. doi:10.1002/3527608958.ch2.

[100] Albert Einstein. Discussion générale des idées nouvelles émises: causalité, déterminisme, probabilité. In *Électrons et photons: Rapports et discussions du cinquième conseil de physique. Tenu à Bruxelles du 24 au 20 octobre 1927 sous les auspices de l'institut international de physique Solvay*, pages 253–256. Gauthier-Villars, Paris, 1928.

[101] Albert Einstein. Quanten-Mechanik und Wirklichkeit. *Dialectica*, 2(3-4):320–324, November 1948. doi:10.1111/j.1746-8361.1948.tb00704.x.

[102] Albert Einstein. Bemerkungen zu den in diesem Bande vereinigten Arbeiten. In Paul Arthur Schilpp, editor, *Albert Einstein als Philosoph und Naturforscher*, chapter III, pages 493–511. W Kohlhammer Verlag, Stuttgart, 1949.

[103] Albert Einstein. On the relativity principle and the conclusions drawn from it. In *The Collected Papers of Albert Einstein*, volume 2: The Swiss Years: Writings, 1900–1909 (English Translation Supplement), chapter 39, pages 252–311. Princeton University Press, 1989. ISBN 0-691-08549-8. Translated by Anna Beck.

[104] Albert Einstein. To Constantin Carathéodory. In Robert Schulmann, Anne J Kox, Michel Janssen, and József Illy, editors, *The Collected Papers of Albert Einstein*, volume 8: The Berlin Years: Correspondence, 1914–1918: Part A: 1914–1917, chapter Document 255, pages 334–335. Princeton University Press, 1998. ISBN 0-691-04849-5. Letter sent on September 6, 1916.

[105] Albert Einstein. To Constantin Carathéodory. In Robert Schulmann, Anne J Kox, Michel Janssen, and József Illy, editors, *The Collected Papers of Albert Einstein*, volume 8: The Berlin Years: Correspondence, 1914–1918: Part A: 1914–1917, chapter Document 284, pages 375–376. Princeton University Press, 1998. ISBN 0-691-04849-5. Letter sent on December 10, 1916.

[106] Albert Einstein. Remarks concerning the essays brought together in this co-operative volume. In Paul Arthur Schilpp, editor, *Albert Einstein Philosopher-Scientist*, volume 7, chapter III, pages 663–688. MJF Books, New York, 2001. ISBN 1-56731-432-5.

[107] Albert Einstein, Boris Podolsky, and Nathan Rosen. Can quantum mechanical description of physical reality be considered complete? *Physical Review*, 47: 777–780, May 1935.

[108] Paul Erker, Mark T Mitchison, Ralph Silva, Mischa P Woods, Nicolas Brunner, and Marcus Huber. Autonomous quantum clocks: how thermodynamics limits our ability to measure time. *preprint arXiv:1609.06704 [quant-ph]*, September 2016. URL `http://arxiv.org/abs/1609.06704`.

[109] Kousha Etessami and Mihalis Yannakakis. On the complexity of Nash equilibria and other fixed points. *SIAM Journal on Computing*, 39(6):2531–2597, January 2010. doi:10.1137/080720826.

[110] Euclid. *Euclid's Elements of Geometry*. Richard Fitzpatrick, 2008. ISBN 978-0-6151-7984-1.

[111] Hugh Everett. "Relative state" formulation of quantum mechanics. *Reviews of Modern Physics*, 29(3):454–462, July 1957. doi:10.1103/RevModPhys.29.454.

[112] Hugh Everett. The theory of the universal wave function. In *The Many-Worlds Interpretation of Quantum Mechanics*, pages 3–140. Princeton University Press, Princeton, 1973.

[113] Julius Farkas. Theorie der einfachen Ungleichungen. *Journal für die reine und angewandte Mathematik*, 124(1):1–27, 1902.

[114] Adrien Feix and Časlav Brukner. Quantum superpositions of "common-cause" and "direct-cause" causal structures. *preprint arXiv:1606.09241 [quant-ph]*, June 2016. URL `http://arxiv.org/abs/1606.09241`.

[115] Adrien Feix, Mateus Araújo, and Časlav Brukner. Causally nonseparable processes admitting a causal model. *New Journal of Physics*, 18(8):083040, August 2016. doi:10.1088/1367-2630/18/8/083040.

[116] Michael R Fellows and Neal Koblitz. Self-witnessing polynomial-time complexity and prime factorization. *Designs, Codes and Cryptography*, 2(3):231–235, September 1992. doi:10.1007/BF00141967.

[117] Paul Feyerabend. Rationalism, relativism and scientific method. *Philosophy in Context*, 6:7–19, 1977. doi:10.5840/philcontext197769.

[118] Paul Feyerabend. *Wissenschaft als Kunst*. Suhrkamp, Frankfurt am Main, 1984. ISBN 978-3-518-11231-1.

[119] Jean-Baptiste Joseph Fourier. Analyse des travaux de l'académie royale des sciences, pendant l'année 1824. Partie mathématique. *Mémoires de l'académie royale des sciences de l'institut de France*, 7, 1827.

[120] Edward Fredkin and Tommaso Toffoli. Conservative logic. *International Journal of Theoretical Physics*, 21(3-4):219–253, April 1982. doi:10.1007/BF01857727.

[121] Stuart J Freedman and John F Clauser. Experimental test of local hidden-variable theories. *Physical Review Letters*, 28:938–941, 1972.

[122] John Friedman and Michael Morris. The Cauchy problem for the scalar wave equation is well defined on a class of spacetimes with closed timelike curves. *Physical Review Letters*, 66(4):401–404, January 1991. doi:10.1103/PhysRevLett.66.401.

[123] John Friedman, Michael S Morris, Igor Dmitriyevich Novikov, Fernando Echeverria, Gunnar Klinkhammer, Kip S Thorne, and Ulvi Yurtsever. Cauchy problem in spacetimes with closed timelike curves. *Physical Review D*, 42(6):1915–1930, September 1990. doi:10.1103/PhysRevD.42.1915.

[124] Komei Fukuda. cdd+-077a. `http://www.inf.ethz.ch/personal/fukudak/cdd_home/cdd.html`, 2003.

[125] Flaminia Giacomini, Esteban Castro-Ruiz, and Časlav Brukner. Indefinite causal structures for continuous-variable systems. *New Journal of Physics*, 18(11): 113026, November 2016. doi:10.1088/1367-2630/18/11/113026.

[126] Vittorio Giovannetti, Seth Lloyd, and Lorenzo Maccone. Quantum time. *Physical Review D*, 92(4):045033, August 2015. doi:10.1103/PhysRevD.92.045033.

[127] Nicolas Gisin. Weinberg's non-linear quantum mechanics and supraluminal communications. *Physics Letters A*, 143(1-2):1–2, January 1990. doi:10.1016/0375-9601(90)90786-N.

[128] Marissa Giustina, Alexandra Mech, Sven Ramelow, Bernhard Wittmann, Johannes Kofler, Jörn Beyer, Adriana E Lita, Brice Calkins, Thomas Gerrits, Sae Woo Nam, Rupert Ursin, and Anton Zeilinger. Bell violation using entangled photons without the fair-sampling assumption. *Nature*, 497(7448):227–230, April 2013. doi:10.1038/nature12012.

[129] Marissa Giustina, Marijn A M Versteegh, Sören Wengerowsky, Johannes Handsteiner, Armin Hochrainer, Kevin Phelan, Fabian Steinlechner, Johannes Kofler, Jan-Åke Larsson, Carlos Abellán, Waldimar Amaya, Valerio Pruneri, Morgan W Mitchell, Jörn Beyer, Thomas Gerrits, Adriana E Lita, Lynden K Shalm, Sae Woo Nam, Thomas Scheidl, Rupert Ursin, Bernhard Wittmann, and Anton Zeilinger. Significant-loophole-free test of Bell's theorem with entangled photons. *Physical Review Letters*, 115(25):250401, December 2015. doi:10.1103/PhysRevLett.115.250401.

[130] Kurt Gödel. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatshefte für Mathematik und Physik*, 38-38(1):173–198, December 1931. doi:10.1007/BF01700692.

[131] Kurt Gödel. An example of a new type of cosmological solutions of Einstein's field equations of gravitation. *Reviews of Modern Physics*, 21(3):447–450, July 1949. doi:10.1103/RevModPhys.21.447.

[132] Kurt Gödel. Eine Bemerkung über die Beziehungen zwischen der Relativitätstheorie und der idealistischen Philosophie. In Paul Arthur Schilpp, editor, *Albert Einstein als Philosoph und Naturforscher*, chapter II. 21, pages 406–412. W Kohlhammer Verlag, Stuttgart, 1951.

[133] Kurt Gödel. Rotating universes in general relativity theory. In Lawrence M Graves, Einar Hille, Paul A Smith, and Oscar Zariski, editors, *Proceedings of the International Congress of Mathematicians*, pages 175–181, Providence, 1952. American Mathematical Society.

[134] Kurt Gödel. Lecture on rotating universes. In Solomon Feferman, John W Jr Dawson, Warren Goldfarb, Charles Parsons, and Wilfried Sieg, editors, *Kurt Gödel: Collected Works*, volume III, pages 269–287. Oxford University Press, New York, 1995. ISBN 978-0-19-507255-6.

[135] Kurt Gödel. A remark about the relationship between relativity theory and idealistic philosophy. In Paul Arthur Schilpp, editor, *Albert Einstein Philosopher-Scientist*, chapter II. 21, pages 555–562. MJF Books, New York, 2001. ISBN 1-56731-432-5.

[136] Kurt Gödel. Gödel to Nagel. In Solomon Feferman, John W Jr Dawson, Warren Goldfarb, Charles Parsons, and Wilfried Sieg, editors, *Kurt Gödel: Collected Works*, volume V, chapter 3, page 147. Clarendon Press, Oxford, 2003. ISBN 978-0-19-850075-9.

[137] Kurt Gödel. Gödel to von Neumann. In Solomon Feferman, John W Jr Dawson, Warren Goldfarb, Charles Parsons, and Wilfried Sieg, editors, *Kurt Gödel: Collected Works*, volume V, chapter 21, pages 372–377. Clarendon Press, Oxford, 2003. ISBN 978-0-19-850075-9.

[138] J Richard Gott. Closed timelike curves produced by pairs of moving cosmic strings: exact solutions. *Physical Review Letters*, 66(9):1126–1129, March 1991. doi:10.1103/PhysRevLett.66.1126.

[139] Branko Grünbaum. *Convex Polytopes*, volume 221 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 2 edition, 2003. ISBN 978-0-387-40409-7. doi:10.1007/978-1-4613-0019-9.

[140] Anil Gupta. Truth and paradox. *Journal of Philosophical Logic*, 11(1):1–60, feb 1982. doi:10.1007/BF00302338.

[141] Anil Gupta and Nuel Belnap. *The revision theory of truth*. MIT Press, Cambridge, 1993. ISBN 978-0-26-252695-1.

[142] Michael J W Hall. *The significance of measurement independence for Bell inequalities and locality*, pages 189–204. Springer International Publishing, Cham, 2016. ISBN 978-3-319-31299-6. doi:10.1007/978-3-319-31299-6_11.

[143] Jonathan J Halliwell, Juan Pérez-Mercader, and Wojciech H Zurek, editors. *Physical Origins of Time Asymmetry*. Cambridge University Press, Cambridge, 1994. ISBN 0-521-56837-4.

[144] Yenjo Han, Lane A Hemaspaandra, and Thomas Thierauf. Threshold computation and cryptographic security. *SIAM Journal on Computing*, 26(1):59–78, February 1997. doi:10.1137/S0097539792240467.

[145] Lucien Hardy. Probability theories with dynamic causal structure: a new framework for quantum gravity. *preprint arXiv:0509120 [gr-qc]*, September 2005. URL `http://arxiv.org/abs/gr-qc/0509120`.

[146] Lucien Hardy. Towards quantum gravity: a framework for probabilistic theories with non-fixed causal structure. *Journal of Physics A: Mathematical and Theoretical*, 40(12):3081–3099, March 2007. doi:10.1088/1751-8113/40/12/S12.

[147] Lucien Hardy. Quantum gravity computers: on the theory of computation with indefinite causal structure. In Wayne C Myrvold and Joy Christian, editors, *Quantum Reality, Relativistic Causality, and Closing the Epistemic Circle*, pages 379–401. Springer Netherlands, Dordrecht, 2009. ISBN 978-1-4020-9106-3. doi:10.1007/978-1-4020-9107-0_21.

[148] Lucien Hardy. Formalism locality in quantum theory and quantum gravity. In Alisa Bokulich and Gregg Jaeger, editors, *Philosophy of Quantum Information and Entanglement*, chapter 3, pages 44–61. Cambridge University Press, Cambridge, 2010. ISBN 978-0-521-89876-8.

[149] Stephen W Hawking. Chronology protection conjecture. *Physical Review D*, 46 (2):603–611, July 1992. doi:10.1103/PhysRevD.46.603.

[150] Werner Heisenberg. Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Zeitschrift für Physik*, 43(3-4):172–198, March 1927. doi:10.1007/BF01397280.

[151] Werner Heisenberg. *Physik und Philosophie*. Hirzel, Stuttgart, 2006. ISBN 978-3-77-761024-5.

[152] Grete Henry-Hermann. Die Kausalität in der Physik. *Studium Generale*, 1(6): 375–383, 1948.

[153] Bas Hensen, Hannes Bernien, Anaïs E Dréau, Andreas Reiserer, Norbert Kalb, Machiel S Blok, Just Ruitenberg, Raymond F L Vermeulen, Raymond N Schouten, Carlos Abellán, Waldimar Amaya, Valerio Pruneri, Morgan W Mitchell, Matthew Markham, Daniel J Twitchen, David Elkouss, Stephanie Wehner, Tim H Taminiau, and Ronald Hanson. Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres. *Nature*, 526(7575):682–686, October 2015. doi:10.1038/nature15759.

[154] Grete Hermann. Die naturphilosophischen Grundlagen der Quantenmechanik. *Die Naturwissenschaften*, 23(42):718–721, October 1935. doi:10.1007/BF01491142.

[155] Grete Hermann. Die naturphilosophischen Grundlagen der Quantenmechanik. In Otto Meyerhof, Franz Oppenheimer, and Minna Specht, editors, *Abhandlungen der Fries'schen Schule. Neue Folge.*, volume 6.2, chapter III, pages 69–152. Verlag "Öffentliches Leben", Berlin, 1935.

[156] Grete Hermann. Zum Vortrag Schlicks. *Erkenntnis*, 6:342–343, 1936.

[157] Grete Hermann. Der Zweifel am Kausalgesetz. In *Die Bedeutung der modernen Physik für die Theorie der Erkenntnis: Drei mit dem Richard Avenarius-Preis ausgezeichnete Arbeiten*, chapter 1, pages 6–14. Verlag von S Hirzel in Leipzig, Leipzig, 1937.

[158] Grete Hermann and Dirk Lumma. The foundations of quantum mechanics in the philosophy of nature. *The Harvard Review of Philosophy*, 7(1):35–44, 1999. doi:10.5840/harvardreview1999715.

[159] Issam Ibnouhsein. *Quantum correlations and causal structures*. PhD thesis, Université Paris Sud, Paris XI, 2015. URL `https://tel.archives-ouvertes.fr/tel-01146097`.

[160] Andrzej Jamiołkowski. Linear transformations which preserve trace and positive semidefiniteness of operators. *Reports on Mathematical Physics*, 3(4):275–278, December 1972. doi:10.1016/0034-4877(72)90011-0.

[161] Marcin Jurdziński. Deciding the winner in parity games is in $\mathsf{UP} \cap \text{co-}\mathsf{UP}$. *Information Processing Letters*, 68(3):119–124, November 1998. doi:10.1016/S0020-0190(98)00150-1.

[162] Chris Kaltwasser. Computing Fibonacci numbers with gates. In *Proceedings of the Research Experience for Undergraduates Program in Mathematics, Oregon State University*, pages 68–83, 1994.

[163] Immanuel Kant. *Kritik der reinen Vernunft*. Hartknoch, Riga, 1781.

[164] Andrei Nikolajewitsch Kolmogorov. Three approaches to the quantitative definition of information. *Problemy Peredachi Informatsii*, 1(1):3–11, 1965.

[165] Kornel Lanczos. Über eine stationäre Kosmologie im Sinne der Einsteinschen Gravitationstheorie. *Zeitschrift für Physik*, 21(1):73–110, December 1924. doi:10.1007/BF01328251.

[166] Gottfried Wilhelm Leibniz and Samuel Clarke. *Correspondence*. Hackett Publishing Company, Inc., Indianapolis, 2000. ISBN 978-0-87220-524-6.

[167] Matt S Leifer and Robert W Spekkens. Towards a formulation of quantum theory as a causally neutral theory of Bayesian inference. *Physical Review A*, 88(5): 052130, November 2013. doi:10.1103/PhysRevA.88.052130.

[168] Jean-Marc Lévy-Leblond. Towards a proper quantum theory (hints for a recasting). *dialectica*, 30(2-3):161–196, September 1976. doi:10.1111/j.1746-8361.1976.tb00727.x.

[169] Ming Li and Paul Vitányi. *An Introduction to Kolmogorov Complexity and Its Applications*. Texts in Computer Science. Springer New York, New York, NY, 2008. ISBN 978-0-387-33998-6. doi:10.1007/978-0-387-49820-1.

[170] Seth Lloyd, Lorenzo Maccone, Raul Garcia-Patron, Vittorio Giovannetti, and Yutaka Shikano. Quantum mechanics of time travel through post-selected teleportation. *Physical Review D*, 84(2):025007, July 2011. doi:10.1103/PhysRevD.84.025007.

[171] Seth Lloyd, Lorenzo Maccone, Raul Garcia-Patron, Vittorio Giovannetti, Yutaka Shikano, Stefano Pirandola, Lee A Rozema, Ardavan Darabi, Yasaman Soudagar, Lynden K Shalm, and Aephraim M Steinberg. Closed timelike curves via post-selection: theory and experimental test of consistency. *Physical Review Letters*, 106(4):040403, January 2011. doi:10.1103/PhysRevLett.106.040403.

[172] Louise Lockwood (Director). Parallel Worlds, Parallel Lives. BBC Scotland, 2007.

[173] Jean-Philippe W MacLean, Katja Ried, Robert W Spekkens, and Kevin J Resch. Quantum-coherent mixtures of causal relations. *preprint arXiv:1606.04523 [quant-ph]*, June 2016. URL `http://arxiv.org/abs/1606.04523`.

[174] Sonia Markes and Lucien Hardy. Entropy for theories with indefinite causal structure. *Journal of Physics: Conference Series*, 306:012043, July 2011. doi:10.1088/1742-6596/306/1/012043.

[175] Serge Massar, Dave Bacon, Nicolas J Cerf, and Richard Cleve. Classical simulation of quantum entanglement without local hidden variables. *Physical Review A*, 63(5):052305, April 2001. doi:10.1103/PhysRevA.63.052305.

[176] Tim Maudlin. Bell's inequality, information transmission, and prism models. *PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association*, 1992(1):404–417, January 1992. doi:10.1086/psaprocbienmeetp.1992.1.192771.

[177] J Clerk Maxwell. On heat engines. In *Theory of Heat*, chapter VIII, pages 138–163. Longmans, Green, and Co., London, 1871.

[178] Nimrod Megiddo and Christos H Papadimitriou. On total functions, existence theorems and computational complexity. *Theoretical Computer Science*, 81(2): 317–324, April 1991. doi:10.1016/0304-3975(91)90200-L.

[179] N David Mermin. Simple unified form for the major no-hidden-variables theorems. *Physical Review Letters*, 65(27):3373–3376, December 1990. doi:10.1103/PhysRevLett.65.3373.

[180] Hermann Minkowski. *Geometrie der Zahlen*. Teubner, Leipzig; Berlin, 1911. ISBN http://gallica.bnf.fr/ark:/12148/bpt6k99643x.

[181] Alberto Montina. Epistemic view of quantum states and communication complexity of quantum channels. *Physical Review Letters*, 109(11):110501, September 2012. doi:10.1103/PhysRevLett.109.110501.

[182] Michael S Morris, Kip S Thorne, and Ulvi Yurtsever. Wormholes, time machines, and the weak energy condition. *Physical Review Letters*, 61(13):1446–1449, September 1988. doi:10.1103/PhysRevLett.61.1446.

[183] Theodore Samuel Motzkin. *Beiträge zur Theorie der Linearen Ungleichungen*. PhD thesis, Universität Basel, 1936.

[184] Michael A Nielsen and Issac L Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, New York, 10 edition, 2009. ISBN 978-0-521-63503-5.

[185] Friedrich Nietzsche. *Die fröhliche Wissenschaft: "La gaya scienza"*. Verlag von E W Fritzsch, Leipzig, 1887.

[186] Friedrich Nietzsche. *Philosophy in the Tragic Age of the Greeks*. Regnery Publishing, Washington DC, 1998.

[187] Friedrich Nietzsche. On truth and lie in a nonmoral sense. In *On Truth and Untruth*, chapter 2, pages 15–62. HarperCollins Publishers, New York, 2010. ISBN 978-0-06-199046-5.

[188] Friedrich Nietzsche. *Die Philosophie im tragischen Zeitalter der Giechen*. Hofenberg, Berlin, 2013. ISBN 978-3-8430-3489-0.

[189] Friedrich Nietzsche. *Über Wahrheit und Lüge im außermoralischen Sinne*. Reclam, Ditzingen, 2015. ISBN 978-3-15-019308-2.

[190] Igor Dmitriyevich Novikov. *Evolution of the Universe*. Cambridge University Press, Cambridge, 1983. ISBN 0-521-24129-4.

[191] Ognyan Oreshkov and Nicolas J Cerf. Operational formulation of time reversal in quantum theory. *Nature Physics*, 11(10):853–858, July 2015. doi:10.1038/nphys3414.

[192] Ognyan Oreshkov and Nicolas J Cerf. Operational quantum theory without predefined time. *New Journal of Physics*, 18(7):073037, July 2016. doi:10.1088/1367-2630/18/7/073037.

[193] Ognyan Oreshkov and Christina Giarmatzi. Causal and causally separable processes. *New Journal of Physics*, 18(9):093020, September 2016. doi:10.1088/1367-2630/18/9/093020.

[194] Ognyan Oreshkov, Fabio Costa, and Časlav Brukner. Quantum correlations with no causal order. *Nature Communications*, 3:1092, October 2012. doi:10.1038/ncomms2076.

[195] Don N Page and William K Wootters. Evolution without evolution: dynamics described by stationary observables. *Physical Review D*, 27(12):2885–2892, June 1983. doi:10.1103/PhysRevD.27.2885.

[196] Abraham Pais. Einstein and the quantum theory. *Reviews of Modern Physics*, 51 (4):863–914, oct 1979. doi:10.1103/RevModPhys.51.863.

[197] Christos H Papadimitriou. *Computational Complexity*. Addison-Wesley Publishing Company, San Diego, 1995. ISBN 0-201-53082-1.

[198] Parmenides. *Vom Wesen des Seienden*. Suhrkamp, Frankfurt am Main, 1989. ISBN 978-3-518-28224-3.

[199] David T Pegg. Quantum mechanics and the time travel paradox. In D Mugnai, A Ranfagni, and L S Schulman, editors, *Time's Arrows, Quantum Measurement and Superluminal Behavior*, page 113. Consiglio Nazionale Delle Richerche, Roma, 2001.

[200] Asher Peres. Incompatible results of quantum measurements. *Physics Letters A*, 151(3-4):107–108, December 1990. doi:10.1016/0375-9601(90)90172-K.

[201] Paolo Perinotti. Causal structures and the classification of higher order quantum computations. *preprint arXiv:1612.05099 [quant-ph]*, December 2016. URL `http://arxiv.org/abs/1612.05099`.

[202] Jacques Pienaar, Timothy C Ralph, and Casey R Myers. Open timelike curves violate Heisenberg's uncertainty principle. *Physical Review Letters*, 110(6):60501, February 2013. doi:10.1103/PhysRevLett.110.060501.

[203] Plato. *The Dialogues of Plato*. Oxford University Press, London, 1892.

[204] Joseph Polchinski. Weinberg's nonlinear quantum mechanics and the Einstein-Podolsky-Rosen paradox. *Physical Review Letters*, 66(4):397–400, January 1991. doi:10.1103/PhysRevLett.66.397.

[205] Sandu Popescu and Daniel Rohrlich. Quantum nonlocality as an axiom. *Foundations of Physics*, 24(3):379–385, March 1994. doi:10.1007/BF02058098.

[206] Huw Price. A neglected route to realism about quantum mechanics. *Mind*, 103 (411):303–336, July 1994. doi:10.1093/mind/103.411.303.

[207] Lorenzo M Procopio, Amir Moqanaki, Mateus Araújo, Fabio Costa, Irati Alonso Calafell, Emma G Dowd, Deny R Hamel, Lee A Rozema, Časlav Brukner, and Philip Walther. Experimental superposition of orders of quantum gates. *Nature Communications*, 6:7913, August 2015. doi:10.1038/ncomms8913.

[208] Timothy C Ralph and Tony G Downes. Relativistic quantum information and time machines. *Contemporary Physics*, 53(1):1–16, January 2012. doi:10.1080/00107514.2011.640146.

[209] Sandra Ranković, Yeong-Cherng Liang, and Renato Renner. Quantum clocks and their synchronisation — the alternate ticks game. *preprint arXiv:1506.01373 [quant-ph]*, June 2015. URL `http://arxiv.org/abs/1506.01373`.

[210] Michael L G Redhead. *Incompleteness, Nonlocality, and Realism: A Prolegomenon to the Philosophy of Quantum Mechanics*. Clarendon Press, Oxford, 1987. ISBN 0-19-824937-3.

[211] Hans Reichenbach. Time order. In *The Philosophy of Space & Time*, chapter 21, pages 135–143. Dover Publications Inc., New York, 1957. ISBN 0-486-60443-8.

[212] Hans Reichenbach. The principle of the common cause. In Maria Reichenbach, editor, *The Direction of Time*, chapter 19, pages 157–167. University of California Press, Berkeley, 1991.

[213] Karl Reinhardt. *Parmenides*. Klostermann, Frankfurt am Main, 1959. ISBN 978-3-465-04144-3.

[214] Wolfgang Rindler. Gödel, Einstein, Mach, Gamow, and Lanczos: Gödel's remarkable excursion into cosmology. *American Journal of Physics*, 77(6):498–510, June 2009. doi:10.1119/1.3086933.

[215] Jérémie Roland and Mario Szegedy. Amortized communication complexity of distributions. In *Automata, Languages and Programming*, pages 738–749. Springer, Berlin, 2009. ISBN 978-3-642-02926-4. doi:10.1007/978-3-642-02927-1_61.

[216] Carlo Rovelli. Loop quantum gravity. *Living Reviews in Relativity*, 1(1):1, December 1998. doi:10.12942/lrr-1998-1.

[217] Giulia Rubino, Lee A Rozema, Adrien Feix, Mateus Araújo, Jonas M Zeuner, Lorenzo M Procopio, Časlav Brukner, and Philip Walther. Experimental verification of an indefinite causal order. *preprint arXiv:1608.01683 [quant-ph]*, August 2016. URL `http://arxiv.org/abs/1608.01683`.

[218] Bertrand Russell. Mathematical logic as based on the theory of types. *American Journal of Mathematics*, 30(3):222, July 1908. doi:10.2307/2369948.

[219] Bertrand Russell. On the notion of cause. *Proceedings of the Aristotelian Society*, 13(1):1–26, June 1913. doi:10.1093/aristotelian/13.1.1.

[220] Daniel Salart, Augustin Baas, Jeroen A W van Houwelingen, Nicolas Gisin, and Hugo Zbinden. Spacelike separation in a bell test assuming gravitationally induced collapses. *Physical Review Letters*, 100(22):220404, June 2008. doi:10.1103/PhysRevLett.100.220404.

[221] Valerio Scarani and Nicolas Gisin. Superluminal influences, hidden variables, and signaling. *Physics Letters A*, 295(4):167–174, March 2002. doi:10.1016/S0375-9601(02)00174-3.

[222] Thomas Scheidl, Rupert Ursin, Johannes Kofler, Sven Ramelow, Xiao-Song Ma, Thomas Herbst, Lothar Ratschbacher, Alessandro Fedrizzi, Nathan K Langford, Thomas Jennewein, and Anton Zeilinger. Violation of local realism with freedom of choice. *Proceedings of the National Academy of Sciences*, 107(46):19708–19713, November 2010. doi:10.1073/pnas.1002780107.

[223] Arthur Schopenhauer. *Die Welt als Wille und Vorstellung: Vier Bücher, nebst einem Anhange, der die Kritik der Kantischen Philosophie enthält*. F A Brockhaus, Leipzig, 1819.

[224] Lynden K Shalm, Evan Meyer-Scott, Bradley G Christensen, Peter Bierhorst, Michael A Wayne, Martin J Stevens, Thomas Gerrits, Scott Glancy, Deny R Hamel, Michael S Allman, Kevin J Coakley, Shellee D Dyer, Carson Hodge, Adriana E Lita, Varun B Verma, Camilla Lambrocco, Edward Tortorici, Alan L Migdall, Yanbao Zhang, Daniel R Kumor, William H Farr, Francesco Marsili, Matthew D Shaw, Jeffrey A Stern, Carlos Abellán, Waldimar Amaya, Valerio Pruneri, Thomas Jennewein, Morgan W Mitchell, Paul G Kwiat, Joshua C Bienfang, Richard P Mirin, Emanuel Knill, and Sae Woo Nam. Strong loophole-free test of local realism. *Physical Review Letters*, 115(25):250402, December 2015. doi:10.1103/PhysRevLett.115.250402.

[225] Peter W Shor. Algorithms for quantum computation: discrete logarithms and factoring. In *Proceedings 35th Annual Symposium on Foundations of Computer Science*, pages 124–134. IEEE Comput. Soc. Press, November 1994. ISBN 0-8186-6580-7. doi:10.1109/SFCS.1994.365700.

[226] Peter W Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM Journal on Computing*, 26(5):1484–1509, October 1997. doi:10.1137/S0097539795293172.

[227] Michael Sipser. On relativization and the existence of complete sets. In *Automata, Languages and Programming*, pages 523–531. Springer-Verlag, Berlin/Heidelberg, October 1982. ISBN 9788578110796. doi:10.1007/BFb0012797.

[228] Ernst Specker. Die Logik nicht gleichzeitig entscheidbarer Aussagen. *dialectica*, 14(2-3):239–246, September 1960. doi:10.1111/j.1746-8361.1960.tb00422.x.

[229] George Svetlichny. Effective quantum time travel. *preprint arXiv:0902.4898 [quant-ph]*, February 2009. URL `http://arxiv.org/abs/0902.4898`.

[230] George Svetlichny. Time travel: Deutsch vs. teleportation. *International Journal of Theoretical Physics*, 50(12):3903–3914, October 2011. doi:10.1007/s10773-011-0973-x.

[231] Alfred Tarski. *Logic, Semantics, Metamathematics*. Clarendon Press, Oxford, 1956.

[232] Max Tegmark. The interpretation of quantum mechanics: many worlds or many words? *Fortschritte der Physik*, 46(6-8):855–862, November 1998. doi:10.1002/(SICI)1521-3978(199811)46:6/8<855::AID-PROP855>3.0.CO;2-Q.

[233] Kip S Thorne. *Black Holes and Time Warps*. W W Norton & Company, New York, 1994. ISBN 978-0-393-31276-8.

[234] Frank J Tipler. Rotating cylinders and the possibility of global causality violation. *Physical Review D*, 9(8):2203–2206, April 1974. doi:10.1103/PhysRevD.9.2203.

[235] Wolfgang Tittel, Jürgen Brendel, Bernard Gisin, Thomas Herzog, Hugo Zbinden, and Nicolas Gisin. Experimental demonstration of quantum correlations over more than 10 km. *Physical Review A*, 57(5):3229–3232, May 1998. doi:10.1103/PhysRevA.57.3229.

[236] Richard Chace Tolman. *The Theory of the Relativity of Motion*. University of California Press, Berkeley, 1918.

[237] Ben F Toner and Dave Bacon. Communication cost of simulating Bell correlations. *Physical Review Letters*, 91(18):187904, October 2003. doi:10.1103/PhysRevLett.91.187904.

[238] Alan M Turing. On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, s2-42(1):230–265, January 1937. doi:10.1112/plms/s2-42.1.230.

[239] Alan M Turing. On computable numbers, with an application to the Entscheidungsproblem. A correction. *Proceedings of the London Mathematical Society*, s2-43(6):544–546, January 1938. doi:10.1112/plms/s2-43.6.544.

[240] Ezio Vailati. *Leibniz and Clarke: A Study of Their Correspondence*. Oxford University Press, Oxford, 1997. ISBN 0-19-511399-3.

[241] Leslie G Valiant. Relative complexity of checking and evaluating. *Information Processing Letters*, 5(1):20–23, May 1976. doi:10.1016/0020-0190(76)90097-1.

[242] Willem Jacob van Stockum. The gravitational field of a distribution of particles rotating about an axis of symmetry. *Proceedings of the Royal Society of Edinburgh*, 57(1938):135–154, September 1938. doi:10.1017/S0370164600013699.

[243] Johann von Neumann. *Mathematische Grundlagen der Quantenmechanik*. Julius Springer, Berlin, 1932.

[244] Paul Watzlawick. *Münchhausens Zopf: oder Psychotherapie und "Wirklichkeit"*. Piper, München, 2005. ISBN 978-3-492-24360-5.

[245] Gregor Weihs, Thomas Jennewein, Christoph Simon, Harald Weinfurter, and Anton Zeilinger. Violation of Bell's inequality under strict Einstein locality conditions. *Physical Review Letters*, 81(23):5039–5043, December 1998. doi:10.1103/PhysRevLett.81.5039.

[246] Hermann Weyl. Über die Zusammenhangsverhältnisse der Welt im Großen. In *Raum Zeit Materie: Vorlesungen über allgemeine Relativitätstheorie*, chapter 33, pages 235–242. Julius Springer, Berlin, 3 edition, 1919.

[247] Hermann Weyl. *Space Time Matter*. Methuen & Co. Ltd., London, 1922.

[248] Hermann Weyl. Elementare Theorie der konvexen Polyeder. *Commentarii Mathematici Helvetici*, 7(1):290–306, December 1934. doi:10.1007/BF01292722.

[249] Stefan Wolf. Nonlocality without counterfactual reasoning. *Physical Review A*, 92(5):052102, November 2015. doi:10.1103/PhysRevA.92.052102.

[250] Stefan Wolf. On bits as fuel. *unpublished*, January 2016. URL `http://cqi.inf.usi.ch/publications/baf.pdf`.

[251] Stefan Wolf. An all-or-nothing flavor to the Church-Turing hypothesis. *preprint arXiv:1702.00923 [quant-ph]*, February 2017. URL `http://arxiv.org/abs/1702.00923`. To appear in the *Proceedings of the 14th Annual Conference on Theory and Applications of Models of Computation*, Bern, 20–22 April, 2017.

[252] Christopher J Wood and Robert W Spekkens. The lesson of causal discovery algorithms for quantum correlations: causal explanations of Bell-inequality violations require fine-tuning. *New Journal of Physics*, 17(3):033002, March 2015. doi:10.1088/1367-2630/17/3/033002.

[253] James Woodward. *Making Things Happen: A Theory of Causal Explanation*. Oxford University Press, Oxford, 2005. ISBN 978-0-19-518953-7.

[254] William K Wootters. "Time" replaced by quantum correlations. *International Journal of Theoretical Physics*, 23(8):701–711, August 1984. doi:10.1007/BF02214098.

[255] Slavoj Žižek (Writer) and Sophie Fiennes (Director). The Pervert's Guide to Cinema. Mischief Films and Amoeba Film, 2006.

[256] Magdalena Zych. *Quantum Systems under Gravitational Time Dilation*. Springer International Publishing, Cham, 2017. ISBN 978-3-319-53191-5. doi:10.1007/978-3-319-53192-2.

Preliminaries and notation

This chapter discusses the mathematical language and the notations used. Some parts of this thesis deal with quantum theory. For that reason, we give a short introduction to quantum information.

The reader who is familiar with quantum information — or not interested— might safely skip this chapter. These preliminaries are held to a minimum and might serve as a reference.

1 Probability theory

We use Latin uppercase letters for random variables, *e.g.*, A. The sample space of a random variable A is denoted by its calligraphic letter $\mathcal{A}$. Latin lowercase letters are used for actual values of random variables. So, a random variable A can take any value $a \in \mathcal{A}$. The probability distribution over a random variable A is expressed by P_A and the probability that the random variable A takes value a is $P_A(a)$ or $P_{A=a}$. The expression $P_{A|B}$ is the conditional probability distribution for A given B. The respective probabilities, then again, are denoted by $P_{A|B}(a,b)$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

If not otherwise stated, finite sample spaces of random variables are assumed to consists of natural numbers from 0 on increasing, *i.e.*, a random variable A with $|\mathcal{A}| = n$ is assumed to have the sample space $\mathcal{A} = \{0, 1, 2, 3, \ldots, n-1\}$. Thus, *binary* random variables are assumed to have the sample space $\{0, 1\}$.

1.1 Matrix and vector representation

Let Q, R be two random variables with the respective sample spaces $\mathcal{Q} = \{q_0, q_1, \ldots\}$, and $\mathcal{R} = \{r_0, r_1, \ldots\}$, and with the probability measures P_Q, P_R.

Definition 31 (State, operation, evolution, composition, value, and probabilty as vectors and matrices)**.** A *state* $\vec{P}_Q$ is a probability vector

$$\vec{P}_Q = \begin{pmatrix} P_Q(q_0) \\ P_Q(q_1) \\ P_Q(q_2) \\ \vdots \end{pmatrix} .$$

An *operation* $\hat{P}_{R|Q}$ is a stochastic matrix

$$\hat{P}_{R|Q} = \begin{pmatrix} P_{R|Q}(r_0,q_0) & P_{R|Q}(r_0,q_1) & P_{R|Q}(r_0,q_2) & \cdots \\ P_{R|Q}(r_1,q_0) & P_{R|Q}(r_1,q_1) & P_{R|Q}(r_1,q_2) & \cdots \\ P_{R|Q}(r_2,q_0) & P_{R|Q}(r_2,q_1) & P_{R|Q}(r_2,q_2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

A state $\vec{P}_Q$ *evolves* through the operation $\hat{P}_{R|Q}$ by the matrix-vector product

$$\vec{P}_R = \hat{P}_{R|Q}\vec{P}_Q .$$

States and operations are *composed* by the Kronecker product $\otimes$.

The *value* r_j that the random variable R can take is modeled by the vector

$$\vec{r}_j^{\,T} = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix} ,$$

where the 1 is placed such that $\vec{r}_j$ represents the deterministic distribution $P_R(r) = \delta_{r,r_j}$. We define $\vec{0}^T$ as the vector $(1,0)$ and $\vec{1}^T$ as the vector $(0,1)$ — they represent both values a binary random variable can take.

For a given state $\vec{P}_R$, the *probability* that the value r_j is found is

$$P_R(r_j) = \vec{r}_j^{\,T}\vec{P}_R .$$

Examples

We discuss two examples to illustrate.

Example 10. Let R and Q be binary random variables, and the operation $\hat{P}_{R|Q}$ be the operation that reads out the value from Q and flips it, *i.e.*,

$$\hat{P}_{R|Q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Now, if the state $\vec{P}_Q$ is

$$\vec{P}_Q = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

then the state after the operation $\hat{P}_{R|Q}$ is

$$\vec{P}_R = \hat{P}_{R|Q}\vec{P}_Q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The probability that state $\vec{P}_R$ takes value 0 is

$$P_R(0) = \vec{0}^T \vec{P}_R = 0.$$

Example 11. Again, assume that the random variable R is binary, and assume that the random variable Q can take four values: $\mathcal{Q} = \{0, 1, 2, 3\}$. We define the operation

$$\hat{P}_{R|Q} = \begin{pmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

This operation outputs the same value as the input on inputs 0 and 1. For the other inputs, the output state are uniformly distributed. So, if the input state is

$$\vec{P}_Q = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix},$$

then the output state is

$$\vec{P}_R = \hat{P}_{R|Q}\vec{P}_Q = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}.$$

If we take two binary random variables Q_1 and Q_2, such that

$$\vec{P}_{Q_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \vec{P}_{Q_2} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$

and give them as input to the operation $\hat{P}_{R|Q}$, where $Q = (Q_1, Q_2)$, then the output is evaluated as

$$\vec{P}_R = \hat{P}_{R|Q}(\vec{P}_{Q_1} \otimes \vec{P}_{Q_2}) = \hat{P}_{R|Q} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

2 Quantum theory

Quantum theory is constructed out of four axioms: the definition of *states*, the definition of *composition of states*, the definition of *evolution*, and the definition of *measurement*. Before we briefly discuss these axioms, we introduce the concept of the Hilbert space and Dirac's *bra-ket* notation. For an introduction to quantum information, we direct the reader to the textbook by Nielsen and Chuang [184].

2.1 The Hilbert space and Dirac's bra-ket notation

A Hilbert space is a complex vector space where the inner product $(\cdot,\cdot)$ is defined. In this thesis, we consider *finite-dimensional* Hilbert spaces only. For two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ we use $\mathcal{L}(\mathcal{H}_A,\mathcal{H}_B)$ to denote the set of linear maps from $\mathcal{H}_A$ to $\mathcal{H}_B$. Likewise, we use $\mathcal{L}(\mathcal{H}_A)$ for the set of liner maps from $\mathcal{H}_A$ to itself. We use d_A to denote the *dimension* of the Hilbert space A. Furthermore, we use $\mathbb{1}$ to denote the identity map, or $\mathbb{1}_A$ to refer to the identity map for system A. Dirac's notation is a notation for the following two operators. For a vector $\psi \in \mathcal{H}_A$, we use $|\psi\rangle \in \mathcal{L}(\mathbb{C},\mathcal{H}_A)$ to describe the linear operator

$$|\psi\rangle : a \mapsto a\psi\,,$$

with $a \in \mathbb{C}$. The operator $|\psi\rangle$ is called *ket*. Its adjoint $(|\psi\rangle^\dagger)$ is called *bra* and is denoted with $\langle\psi| \in \mathcal{L}(\mathcal{H}_A,\mathbb{C})$. The bra operator acts as

$$\langle\psi| : \phi \mapsto (\psi,\phi)\,.$$

Sometimes we amend bras or kets with subscripts to indicate quantum system, *i.e.*, $|\psi\rangle_A$ is an operator from the set $\mathcal{L}(\mathbb{C},\mathcal{H}_A)$.

By combining a *bra* $\langle\psi| \in \mathcal{L}(\mathcal{H}_A,\mathbb{C})$ with a *ket* $|\phi\rangle \in \mathcal{L}(\mathbb{C},\mathcal{H}_A)$ we get a *bra-ket*:

$$\langle\psi|\cdot|\phi\rangle = \langle\psi|\phi\rangle \in \mathcal{L}(\mathbb{C})\,,$$

or a *ket-bra*:

$$|\phi\rangle\langle\psi| \in \mathcal{L}(\mathcal{H}_A)\,.$$

The *trace* of a matrix M is the sum of its diagonal entries and is expressed by $\operatorname{Tr} M$.

2.2 Axioms

Axiom 1 (Quantum state)**.** A *pure quantum state* of a quantum system A is a ket operator $|\psi\rangle$ constructed from a unit length vector $\psi \in \mathcal{H}_A$. The set of all pure states of a system A is denoted by $\mathcal{P}(\mathcal{H}_A) \subset \mathcal{L}(\mathbb{C},\mathcal{H}_A)$. If we wish to express the state of a subsystem, or incorporate ignorance in a description of a quantum state, then this state is not necessarily a *pure* one, but might be *mixed*. A *mixed quantum state* (also called density operator) of a quantum system A is an operator ρ form the set $\mathcal{L}(\mathcal{H}_A)$ which is constrained to be positive, *i.e.*, $\rho \geq 0$, and to have unit trace: $\operatorname{Tr}\rho = 1$. By $\mathcal{S}(\mathcal{H}_A) \subset \mathcal{L}(\mathcal{H}_A)$ we refer to the set of all density operators from the set $\mathcal{L}(\mathcal{H}_A)$. A mixed state ρ is a convex combination of orthogonal pure states $|\phi_1\rangle, |\phi_2\rangle, \ldots$ with the respective probabilities $p_1, p_2, \ldots$:

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|\,.$$

For the two-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$ we use

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

as the standard basis.

Axiom 2 (Composition)**.** Hilbert spaces are composed with the tensor product: $\otimes$.

Axiom 3 (Evolution)**.** A *unitary* operator U is an element in $\mathcal{L}(\mathcal{H}_A)$ such that

$$U^\dagger U = UU^\dagger = \mathbb{1}_A,$$

and where

$$U\mathbb{1}_A = \mathbb{1}_A.$$

The first condition ensures reversibility and the second conditions preserves the inner product.

By this axiom, a unitary U maps orthogonal pure states to orthogonal pure states. A unitary is applied to a pure state by $U|\phi\rangle$, and to a mixed state by $U\rho U^\dagger$.

Axiom 4 (Measurement)**.** A measurement is defined as a Hermitian operator called *observable* $O = \sum_{x\in\mathcal{X}} P_x$, where $\mathcal{X}$ represents the set of possible measurement outcomes and where P_x are orthogonal projectors. The probability that a measurement of state ρ yields the result x is given by

$$P_X(x) = \mathrm{Tr}\left(P_x\rho\right).$$

The most general transformations from quantum states to quantum states

$$\mathcal{S}(\mathcal{H}_A) \to \mathcal{S}(\mathcal{H}_B)$$

are completely-positive trace-preserving (CPTP) maps. Such an evolution can be described as a unitary evolution if one incorporates auxiliary quantum systems. A measurement can be seen as a completely-positive (CP) map ($\mathcal{S}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$), where the trace of the output of the map is the probability for a particular CP map to be applied.

2.3 The CJ isomorphism and composition of Choi maps

The Choi-Jamiołkowski isomorphism [72, 160] is an isomorphism between CP maps and linear operators on the tensor-product space. Given some CP map

$$\varepsilon : \mathcal{S}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B),$$

the *Choi map*, as the map in the tensor-product space is called, is

$$M^{\varepsilon} = (\mathbb{1}_A \otimes \varepsilon)\,|\Phi\rangle\langle\Phi|_{A,A'} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)\,,$$

where

$$|\Phi\rangle_{A,A'} = \sum_i |i\rangle_A \otimes |i\rangle_{A'}$$

is the unnormalized maximally entangled state, and where the system A' is isomorphic to A. Given some Choi map $M \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, the corresponding CP map is

$$\begin{aligned}\varepsilon^E : \mathcal{S}(\mathcal{H}_A) &\to \mathcal{L}(\mathcal{H}_B)\,,\\ \rho_A &\mapsto Tr_A\big(\big(\rho_A^T \otimes \mathbb{1}_B\big)M\big)\,,\end{aligned}$$

where the superscript T denotes the transposition.

For a Choi map $M \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ to represent a CPTP map, the following condition must hold:

$$\mathrm{Tr}_B\, M = \mathbb{1}_A\,.$$

For M to represent a CP map which is not CPTP

$$\mathrm{Tr}_B\, M < \mathbb{1}$$

has to hold. The probability that some CP map M_k is applied to a input state ρ_A is

$$p_k = \mathrm{Tr}\big(\big(\rho_A^T \otimes \mathbb{1}_B\big)M_k\big)\,.$$

Since the probabilities have to sum to unity, it is clear that a CPTP map M can be written the sum of a collection of CP maps $\{M_k\}_{\mathcal{K}}$:

$$1 = \sum_{k\in\mathcal{K}} p_k = \sum_{k\in\mathcal{K}} \mathrm{Tr}\big(\big(\rho_A^T \otimes \mathbb{1}_B\big)M_k\big) = \mathrm{Tr}\big(\big(\rho_A^T \otimes \mathbb{1}_B\big)M\big)\,.$$

We briefly describe how to compose two Choi maps in sequence — which is called the *link product*. For a full introduction, we refer the reader to the article by Chiribella, D'Ariano, and Perinotti [68]. Let M, N be to Choi maps where the system A is (part of) the output from M as well as (part of) the input to N. The composition of these two maps yields the Choi map

$$C = A \star B = \mathrm{Tr}_A\big(M^{T_A} N\big)\,,$$

where the superscript T_A denotes a transposition of the system A. Furthermore, we omitted possible identity operators in the above equation. Choi maps, like other quantum transformations, are composed in parallel by the tensor product.

3 Polytopes

A d-dimensional convex polytope $\mathcal{P}$ (or simply *polytope*) is a convex set of points in $\mathbb{R}^d$ with a *finite* number of extremal points. An extremal point e of $\mathcal{P}$ is a point that cannot be written as a convex combination of other points from the polytope, *i.e.*, if $e = \lambda x + (1-\lambda)y$ with $x, y \in \mathcal{P}$ and $0 < \lambda < 1$, then $e = x = y$. Every polytope can be represented by the set of its *extremal points*. Then, all points of the polytope can be expressed by a convex mixture of these extremal points. Such a representation is called *V-representation*, where V stands for *vertex*. Another way to represent a polytope is by the set of *half-spaces*: The intersection of these half-spaces is the polytope. This representation is called *H-representation*. To switch from the H-representation to the V-representation one can use algorithms like the double-description method.[1] The idea of that method is the following. The pair (A, R) of matrices is called *double description pair* if

$$Ax \geq 0 \Longleftrightarrow \exists \lambda \geq 0 : x = R\lambda \,.$$

The columns of A are the half-spaces, and the rows of R are the extremal points of a polytope $\mathcal{P}$. By Minkowski's theorem [180] and by Weyl's theorem [248], a polytope can be represented either by A or by R. The double-description method is an iterative algorithm to compute R from A, where one starts with a few half-spaces (a few columns from A) and in every iteration, the number of half-spaces is increased. The Fourier-Motzkin [119, 183] elimination algorithm is more general than the double-description method. This method has been implemented in the software cdd+ [124].

The inverse problem, which is the dual of the double-description method, can be solved by a convex-hull algorithm [21, 23] which is essentially the *dual* of the double-description method. This follows from Faraks' [113] lemma which states that (A, R) is a double-description pair if and only if (R^T, A^T) is a double-description pair. By that lemma, one can simply transpose the vectors to obtain a convex-hull algorithm.

For a textbook on convex polytopes see Grünbaum [139].

[1] This method was independently discovered by Chernikova [67].

Epilogue: Anti-realism

After having completed this thesis, I[1] feel inclined to continue by going back to the *pre-Socratic time*, to start a discussion on what is called *anti-realism* [25, 96]. At this point, we would like to warn the reader: What is going to come is a personal account, sometimes vague, and potentially displeasing to some readers.

Anti-realism in epistemology is often linked to the notion of "truth." An anti-realist attitude towards the notion of "truth," is that any form of "absolute truth" is *denied*. In a weaker form, it can also be understood as the attitude that not *transcendental* "truth" exists, beyond any perception or thought process. Furthermore, any true (or wrong) statement must be verifiable in principle [25]. This does not mean that true statements cannot be formed, or shown. Rather, the notion of "truth" is tightly bound to the context, *e.g.*, formal system, language, cultural-historic context, *etc.* Such contexts, however, are built up by governments, societies, research groups, or even individuals, giving a "subjective" taste.

In epistemology, anti-realism could be defined as follows:

Definition 32 (Anti-realism, as defined in Reference [25])**.** What we actually can think or say about the world depends fundamentally on our perspective on the world.

Opposed to that, the *realist* point of view asserts that "truths" exist that are independent of the viewer, language, *etc.*, which are a property of "reality" (which exists independent of the observer, see Section 2.1.1). So, the anti-realist's point of view is that "truths" depend on our perspective and abilities, whilst the realist's point of view is that "truth" is "out there" independent of any perspective or observer.

We start with a discussion on *Parmenides'* didactic poem, let Nietzsche talk about Parmenides, and list some *anti-realists* statements. The purpose of that is to briefly illustrate anti-realism, to thereafter combine that view with the beginning of this thesis: the motivations.

Here, this endeavour is taken because *anti-realism* is thoroughly an intriguing concept, and "there was never any argument to show that the modern and more totalitarian

[1]These are the only four occurrences of singular first person pronouns in this thesis to refer to the author; I am not sure whether anyone else is inclined or even interested to these upcoming thoughts, yet, since this my thesis, I take that freedom to follow this impulse.

notion of truth has advantages, and what the advantages are" [117]. In the following, longer quotes are placed, because the originals are — in our opinion — worth reading.

4 Parmenides of Elea

Parmenides was a Greek philosopher from Elea (Elea was an ancient city whose ruins are found next to the town Aseca, province of Salerno, Italy) who lived around −500. In his poem [86] (unfortunately fragments thereof only survived, see also References [198, 213]), he describes an insight he had possibly at "einen Moment der allerreinsten, durch jede Wirklichkeit ungetrübten und völlig blutlosen Abstraktion"[2] [188]. In an artistic form, Parmenides describes himself as visiting the goddess Dike, who passes the "knowledge" of the "being" to him. After welcoming him, Dike says: "It is indifferent to me whence I begin, for to that place I shall come back again" [86]. By this, Dike indicates that a *sole* line of thought is possible, whereas the alternative is "wholly without report." Thereafter, Dike describes the two lines of thought: The path where one treats *presences* as *being*, and *absences* as *not being*; and the other path where *absences* are also considered as *being*. On the first path, one considers the "being" as existent and the "not being" as inexistent. In other words, the "being" *is*, and the "void" *is not*. On the other path, the "being" and the "void" *are*, yet, as Dike tells Parmenides, "you can neither know what is not (for it is impossible) nor tell of it." So, she concludes, this alternative path is *impossible*; for how can something that *is not* be? By this oldest occurrence of a proof of contradiction in recorded history, she indicates that the other path is the only one. From this point on, Dike proceeds by deriving the properties of "what is" (again, with proofs of contradiction):

- The "being" did not come into existence. Out of what should it have been created?
- The "being" is indestructible, for it cannot cease existence and be transformed into the "void" — there is no "void."
- The "being" is *one* thing as otherwise it would have *parts*. But to have parts, there must be some distinction among the parts (absence in one, presence in the other).
- *etc.*

To cut a long story short[3], the "being" has no other attribute than that *it is*.

Our *appearance* of the world, however, has to be distinguished from the "being;" we, *e.g.*, have the perception of space and time. So, how can our perception be put in

[2]As translated in Reference [186]: "a moment of purest absolutely bloodless abstraction, unclouded by any reality."

[3]Plato [203] dedicated a whole book to these derivations, where he lets Parmenides and his pupil Zeno debate with the young Socrates.

agreement with the "being?" Here, Dike answers that the appearance emerges because the "mortals" (note that Dike is a goddess) are "two-headed, ... [to] whom this has been accepted as both being and not being the same and not the same." For the mortals,

> "all those things will be a name [...], confident that they are real, suppose to be coming to be and perishing, to be and not to be, and to change their place and alter their bright aspect to dark and from dark to bright. [...] For they resolved to name two [f]orms (of which it is wrong to name only one, wherein men have gone astray) [...]."

To conclude, the "being" has no attributes other than it is, and experience is illusion arising from being two-headed: We name the "void," yet the "void" does not exist, and by that we obtain the illusions of objects.

5 Nietzsche's view on Parmenides

We briefly consult Nietzsche [188], who probably gave one of the most accurate (definitely, pleasing) description of Parmenides' poem. Nietzsche describes the *young* Parmenides as being suspicious of separating the world into a world which *is*, and another which *is not*.

> "Verglich er zum Beispiel Licht und Dunkel, so war die zweite Qualität ersichtlich nur die *Negation* der ersten; [...] so daß vor seinem Blicke sich unsre empirische Welt in zwei getrennte Sphären schied, in die der positiven Eigenschaften — mit einem lichten, [...] Charakter — und in die der negativen Eigenschaften."[4]

Nietzsche's Parmenides concludes that both "spheres" must interact in order for something to come into existence. But, one day, as Nietzsche writes,

> "prüfte er seine beiden zusammenwirkenden Gegensätze, deren Begierde und Haß die Welt und das Werden konstituiert, das Seiende und das Nichtseiende, die positiven und die negativen Eigenschaften — und er blieb plötzlich bei dem Begriffe der negativen Eigenschaft, des Nichtseienden, mißtrauisch hängen. Kann denn etwas, was nicht ist, eine Eigenschaft sein? Oder prinzipieller gefragt: kann denn etwas, was nicht ist sein?"[5]

[4]As translated in Reference [186]: "Comparing, for example, light and dark, he found the latter obviously but the *negation* of the former. [...] Thus before his gaze our empirical world divided into two separate spheres, the one characterized by light, [...], and the other by the opposite, negative qualities. The latter really express only the lack, the absence of the former, positive ones."

[5]As translated in Reference [186]: "he tested his two interactive contradictories, whose mutual desire and hatred constitute the world and all coming-to-be. He tested the existent and the nonexistent, the positive and the negative properties — and suddenly he found that he could not get past the concept of a negative quality, the concept of non-existence. Can something which is not be a quality? Or, more basically, can something which is not, be?"

In that, Nietzsche sees Parmenides to come of age from "merely participat[ing] in a universal crime against logic," to a radical world view where

> "jede Mühe verschwendet [ist], die man sich mit dieser erlogenen, durch und durch nichtigen und durch die Sinne gleichsam erschwindelten Welt gibt."[6]

Nietzsche as an anti-realist, criticizes Parmenides:

> "[Parmenides] erschloß, daß es existieren müsse: ein Schluß der auf der Voraussetzung beruht, daß wir ein Organ der Erkenntnis haben, das ins Wesen der Dinge reicht und unabhängig von der Erfahrung ist. Der Stoff unseres Denkens ist nach Parmenides gar nicht in der Anschauung vorhanden, sondern wird anderswoher hinzugebracht, aus einer außersinnlichen Welt, zu der wir durch das Denken einen direkten Zugang haben."[7]

Thus, as, Nietzsche sees it, Parmenides is not an anti-realist in the above sense. Rather, in his poem, Parmenides aims at making absolute statements about "what there is," even though it is inaccessible and independent of the observer. For Nietzsche, this concept of "being" is empty:

> "Die Worte sind nur Symbole für die Relationen der Dinge untereinander und zu uns und berühren nirgends die absolute Wahrheit: und gar das Wort 'Sein' bezeichnet nur die allgemeinste Relation, die alle Dinge verknüpft [...]. Durch Worte und Begriffe werden wir nie hinter die Wand der Relationen, etwa in irgendeinen fabelhaften Urgrund der Dinge, gelangen, und selbst in den reinen Formen der Sinnlichkeit und des Verstandes, in Raum, Zeit und Kausalität gewinnen wir nichts [...]. Es ist unbedingt für das Subjekt unmöglich, über sich selbst hinaus etwas sehen und erkennen zu wollen, so unmöglich, daß Erkennen und Sein die sich widersprechendsten aller Sphären sind."[8]

[6]As translated in Reference [186] "[a]ll effort spent upon this false deceitful world which is futile and negligible, faked into a lying existence by the senses [...] therefore wasted [is]."

[7]As translated in Reference [186]: "[Parmenides] concluded its existence from the fact that he was able to think it. This is a conclusion which rests on the assumption that we have an organ of knowledge which reaches into the essence of things and is independent of experience. The content of our thinking, according to Parmenides, is not present in sense perception but is an additive from somewhere else, from an extra-sensory world to which we have direct access by means of our thinking."

[8]As translated in Reference [186]: "Words are but symbols for the relations of things to one another and to us; nowhere do they touch upon absolute truth. Above all, the word 'being' designates only the most general relationship which connects all things [...]. Through words and concepts we shall never reach beyond the wall of relations, to some sort of fabulous primal ground of things. Even in the pure forms of sense and understanding, in space, time and causality, we gain nothing that resembles an eternal verity. It is absolutely impossible for a subject to see or have insight into something while leaving itself out of the picture, so impossible that knowing and being are the most opposite of all spheres."

At the pre-Socratic time, as Nietzsche writes, Parmenides' approach might have been acceptable. After Kant, however, any attempt to derive the "absolute" from a subjective concept is "certainly reckless ignorance" [186].

6 Anti-realists' statements in other fields

We present some statements that go in this direction in order to make the subject more clear by demonstration. This *denial of absolute truth* is a building block of Nietzsche's philosophy, which is prominent in his work *Über Wahrheit und Lüge im außermoralischen Sinne* [189]. Nietzsche wrote this work when he was 29, but it got published only posthumously. We demonstrate his standpoint with the following quote.

> "Wenn ich die Definition des Säugethiers mache und dann erkläre, nach Besichtigung eines Kameels: Siehe, ein Säugethier, so wird damit eine Wahrheit zwar an das Licht gebracht, aber sie ist von begränztem Werthe, ich meine, sie ist durch und durch anthropomorphisch und enthält keinen einzigen Punkt, der 'wahr an sich', wirklich und allgemeingültig, abgesehen von dem Menschen, wäre."[9]

In philosophy of science, Feyerabend [118] can be considered as an anti-realist:

> "Untersucht man nämlich, was ein bestimmter Denkstil unter ['Wahrheit' und 'Wirklichkeit'] versteht, dann trifft man nicht auf etwas, was jenseits des Denkstils liegt, sondern auf seine eigenen grundlegenden Annahmen: Wahrheit ist, was der Denkstil sagt, daß Wahrheit sei. So war es einmal wahr, daß die griechischen Götter existierten, aber heute ist das für viele Menschen Unsinn."[10]

Here, we see that Feyerabend considers the notion of "truth" and "reality" as attached to a *school of thought*. So, it is impossible to transcend that *school of thought* to arrive at other notions — a theory, a *school of thought*, already defines what it understands under "truth" and "reality."

Then again, the psychologist Watzlawick can be understood as being as favoring anti-realism (or *instrumentalist*):

[9] As translated in Reference [187] "If I give a definition of 'mammal' and then, after inspecting a camel, declare, 'Behold, a mammal,' a truth has indeed been brought to light, but one of limited value, by which I mean it is thoroughly anthropomorphic and contains not a single point that would be 'true in itself,' real and universally valid, apart from man."

[10] "If one examines what a particular school of thought considers as 'truth' and 'reality,' then one does not find something beyond the school of thought, instead, one finds its own fundamental assumptions: Truth is what the school of thought says it is. So, it was once true that the Greek gods existed, but today, for most people this is nonsense."

> "Aber der Zweck wissenschaftlicher Forschung ist und kann nicht die Entdeckung der Wahrheit sein. Für die ewige Wahrheit ist kein Platz in der Wissenschaft — vor allem nicht auf einem Gebiet, das so unerfaßbar ist wie das seelische und geistige Erleben des Menschen. Das einzige brauchbare Kriterium ist die größere *Wirksamkeit* eines Ansatzes im Vergleich zu einem anderen" [244].

Finally, the psychoanalyst Žižek [255] notes that we are fooled to believe in some "reality," even in the case where we see the "stage machinery" behind it:

> "The mystery is that even if we know that it's only staged, that it's a fiction, it still fascinates us. That's the fundamental magic of it. You witness a certain seductive scene, then you are shown that it's just a fake, stage machinery behind, but you are still fascinated by it. Illusion persists. There is something real in the illusion, more real than in the reality behind it. [...] It is that rather, in a way, there is more truth in this appearance. Appearance has an effectivity, a truth of its own."

7 Esquisse of a synthesis

First, Nietzsche and Parmenides can be brought closer by the following. Observe that the critique of Nietzsche is that Parmenides relies on some "organ of knowledge which reaches into the essence of things." Then again, it is *not* Parmenides who talks to us, but the goddess Dike. In that sense, Dike is the "organ of knowledge." However, form a "two-headed" point of view, this knowledge is inaccessible; very much in the spirit of Kant [163]. So, on the one hand Parmenides describes "what there is," but on the other, he denies its accessibility.

To connect the concept of anti-realism with this thesis, one cold see the sketched approaches of *relative*, *retro*, and *emergent causality* (see Section 2.1.5), as valid approaches to science without any further request to capture some *underlying truth* — a kind of toy models (after all, any theory is not more than a toy model; a theory is *not* "reality"). The conditions for a statement to be true would in that case be given by the model itself. Yet, these sketched approaches are a response to the requirement of *local realism*. But here, *local realism* is to be understood as a condition that fits our *classical way of thinking*; a condition we are comfortable with. Some generations to the future — one could have hoped that this was the case already now — quantum theory could become *easier to "understand"* when compared to classical ones. So, the quest for a *local realistic* theory to describe quantum correlations is a quest that clearly leads to new insights, but it should not be seen in a too serious way: *local realism* is just some definition we made and we are comfortable with now.

Anti-realism can also be seen as an *extreme case* of the approach titled *emergent causality*. In the latter, classical qualities, from bits to space and time, would be the product of *perception* (a concept not clear at all), without presupposing that these qualities exist *a priori*. In the former, the conditions for what counts as *true* is emergent. The *emergent causality* approaches can also — in a wide sense — be identified with *Parmenides'*: The "is" is some "blob" without classical qualities, and by being *two-headed*, the classical qualities arise. One could — by allowing one self to broaden the sense even more — even identify *two-headedness* with *coarse graining*. In the description of Parmenides, two-headedness means that one attributes "nothingness" to what is, and *vice versa*. In the process of *coarse graining*, then again, some quantities are *ignored*.

A formulation of quantum theory that fits Parmenides' model in some way,[11] is the one by Page and Wootters (see References [195, 254]). There, the "blob" is the *timeless wavefunction*, and any observation is only an observation *relative* to "clock time." In more detail, one can construct a quantum state over multiple systems, where one system is called "clock." Now, if one entangles the "clock" with the rest, then it is possible to interpret the evolution of the "rest" as relative to the "clock time:" the state of the "clock" system. This can be understood as an extension of *Everett's [111, 112] relative state formalism*: where "clock time" has been incorporated into the description. This entangled state is not evolving any more, yet, a *subjective* evolution takes place — an intriguing idea.

Because we are (partially) comfortable with the ongoing school of thought, and definitely *fascinated* by quantum theory, we conduct research in this field. For large parts it is this *tension* of quantum theory against the traditional school of thought that drives research in this field; the tension between quantum concepts (quantum effects) and their incompatibility with what we are so much used to. By consulting Feyerabend [118]:

> "Die Wahl eines Stils, einer Wirklichkeit, einer Wahrheitsform, Realitäts- und Rationalitätskriterien eingeschlossen, ist die Wahl von Menschenwerk. Sie ist ein *sozialer Akt* [...],"[12]

we see that science is also a *sociological act*.

By stating this anti-realist point of view, we do not (1) claim ourselves anti-realists, nor (2) trivialize theoretical problems. Against (1): no one knows; and (2): we chose to work in this field for some reason. We end this epilogue: An anti-realist point of view does not at all render scientific work less important, but allows one to approach science in a more playful, less "truth-seeking" way.

[11] Parmenides was quite vague, so in *retrospect* many things can be understood as following his poem. Here, we just focus on the emergence of qualities from a state of affairs where these qualities are inexistent.

[12] "The choice of a style, an objectivity, a form of truth, including conditions for reality and rationality, is the choice of people. It is a *social act*,"