The Classical-Communication Rate of Quantum Resources

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***Abstract*—Quantum theory is, in some way, "non-classical." For instance, the behavior of entangled systems (i.e., shared quantum information) under measurements cannot, in general, be explained by shared classical information. With classical *communication*, on the other hand, both the correlations entanglement leads to as well as quantum channels can be reproduced in principle. Here, crucial questions are whether the required communications is *finite*; if so, then its exact amount is related to the "degree of non-classicality" of the quantum primitive. We apply information-theoretic results such as the reverse Shannon theorem for determining the required communication in the asymptotic limit.**

The communication complexity of a quantum channel is the minimal amount of classical communication required for classically simulating the process of preparation, transmission through the channel, and subsequent measurement of a quantum state. At present, only little is known about this quantity. Our generic procedure allows for systematically evaluating the communication complexity of channels in any general probabilistic theory, in particular quantum theory. The procedure is constructive and provides the most efficient classical protocols. We illustrate it by evaluating the communication complexity of a noiseless quantum channel with some finite sets of quantum states and measurements.

As a second application, we determine the classical-communication rate required for the simulation of the behavior under measurements of entangled states. Here, the communication cost can be directly interpreted as the "non-classicality" of the correlation. A particular example is the simulation of non-maximally entangled pure qubit pairs, where we find the required communication rate to behave monotonously with the strength of the entanglement. For different measures of non-locality, such as the number of required non-local (PR) boxes, another behavior had been reported for the single-shot scenario.

I. Introduction

We present a general method, based on classical channel-coding results, for constructively determining the cost of realizing quantum resources in terms of classical communication. We illustrate our procedure for quantum channels and sketch a further application, entanglement simulation. We anticipate even broader applicability.

Quantum communication has proved to be much more powerful than its classical counterpart. Indeed, quantum channels can provide an exponential saving of communication resources in some distributed computing problems [1], where the task is to evaluate a function of data held by two or more parties. A natural measure of power of quantum communication is provided by the communication complexity of a quantum channel, which is defined as the minimal amount of classical communication required for classically simulating the process of preparation, transmission through the channel, and subsequent measurement of a quantum state. Indeed, it is clear that a quantum channel cannot replace an amount of classical communication greater than its communication complexity. Thus, this quantity sets an ultimate limit to the power of quantum communication in a two-party scenario in terms of classical resources.

At present, only little is known about the communication complexity of quantum channels. Toner and Bacon proved that two classical bits are sufficient to simulate the communication of a single qubit [2]. In the case of parallel simulations, the communication can be compressed so that the asymptotic cost per simulation is about 1.28 bits [3]. Simulating the communication of n qubits requires an amount of classical communication greater than or equal to $2^n - 1$ bits [4]. However, no upper bound is known.

In this paper, we present a general procedure for systematically evaluating the communication complexity of channels in any general probabilistic theory, in particular quantum theory. The procedure relies on the reverse Shannon theorem [5] and a strategy discussed in Refs. [3], [6]. We illustrate this procedure by evaluating the communication complexity of a noiseless quantum channel with some finite sets of quantum states and measurements.

II. Method

A protocol simulating a quantum channel actually simulates a process of preparation, transmission through the channel and subsequent measurement of a quantum state. For the sake of simplicity, we will focus on noiseless quantum channels, but the following discussion can be easily generalized to any probabilistic theory, as pointed out later. The simulated quantum scenario is as follows. A party, say Alice, prepares n qubits in some quantum state $|\psi\rangle$. Then, she sends the qubits to another party, say Bob. Finally, Bob generates an outcome by performing a measurement $\mathcal{M} = \{\hat{E}_1, \hat{E}_2, \dots\}$, where $\hat{E}_i$ are positive semidefinite self-adjoint operators labeling events of the measurement $\mathcal{M}$. In a classical simulation, the quantum channel between Alice and Bob is replaced by classical communication. A classical protocol is as follows. Alice sets a variable, say k, according to a probability distribution $\rho(k|y,\psi)$ that depends on the quantum state $|\psi\rangle$ and, possibly, a random variable y shared with Bob. Thus, there is a mapping from the quantum state to a probability distribution of k,

$$|\psi\rangle \xrightarrow{y} \rho(k|y,\psi). \tag{1}$$

Alice sends k to Bob, who simulates a measurement $\mathcal{M}$ by generating an outcome $\hat{E}_w$ with a probability $P(w|k,y,\mathcal{M})$. The protocol exactly simulates the quantum channel if the probability of $\hat{E}_w$ given $|\psi\rangle$ is equal to the quantum probability, that is, if

$$\sum_k \int dy P(w|k,y,\mathcal{M})\rho(k|y,\psi)\rho(y) = \langle\psi|\hat{E}_w|\psi\rangle, \tag{2}$$

where $\rho(y)$ is the probability density of the random variable y. Let us denote by $\rho(k|y) \equiv \int d\psi \sum \rho(k|y,\psi)\rho(\psi)$ the marginal conditional probability of k given y. As defined in Ref. [6], the communication cost, say $\mathcal{C}$, of the classical simulation is the maximum, over the space of distributions $\rho(\psi)$, of the Shannon entropy of the distribution $\rho(k|y)$ averaged over y, that is,

$$\mathcal{C} \equiv \max_{\rho(\psi)} H(K|Y), \tag{3}$$

where $H(K|Y) \equiv -\int dy \rho(y) \sum_k \rho(k|y) \log_2 \rho(k|y)$.

This definition relies on the Shannon coding theorem, as discussed in Ref. [6]. We define the *communication complexity* (denoted by $\mathcal{C}_{min}$) of a quantum channel as the minimal amount of classical communication $\mathcal{C}$ required by an exact classical simulation of the quantum channel, given any measurement $\mathcal{M}$. Let $\mathbf{S} \equiv \{|\psi_1\rangle,\dots,|\psi_S\rangle\}$ and $\mathbf{M} \equiv \{\mathcal{M}_1,\dots,\mathcal{M}_M\}$ be a set of S quantum states and M measurements, respectively. We define the communication complexity, say $\mathcal{C}_{min}(\mathbf{G})$, of the quantum game $(\mathbf{S},\mathbf{M}) \equiv \mathbf{G}$ as the minimal amount of classical communication required to simulate the quantum channel with the restriction that the quantum states and the measurements are elements of $\mathbf{S}$ and $\mathbf{M}$, respectively.

Let us consider the case of R quantum channels. In a general parallel simulation, the communicated variable k is generated according to a probability distribution $\rho(k|y,\psi^1,\psi^2,\dots,\psi^R)$ depending on the whole set of R prepared quantum states $|\psi^1\rangle,\dots,|\psi^R\rangle$ (superscripts will always label channels simulated in parallel). Thus, the single-shot map (1) is replaced by

$$\{|\psi^1\rangle,\dots,|\psi^R\rangle\} \xrightarrow{y} \rho(k|y,\psi^1,\psi^2,\dots,\psi^R). \tag{4}$$

The asymptotic communication cost, say $\mathcal{C}^{asym}$, is equal to $\lim_{R\to\infty} \mathcal{C}^{par}/R$, $\mathcal{C}^{par}$ being the cost of the parallelized simulation. The definition of $\mathcal{C}^{par}$ is similar to that of $\mathcal{C}$, with the difference that the maximization is made over the space of the distributions $\rho(\psi^1,\dots,\psi^R)$. We define the asymptotic communication complexity, $\mathcal{C}_{min}^{asym}$, of a quantum channel as the minimal asymptotic communication cost required for simulating the channel. The asymptotic communication complexity of the game $\mathbf{G}$ is similarly defined.

Given a game $\mathbf{G} = (\mathbf{S},\mathbf{M})$, let $\mathbf{w} = \{w_1,\dots,w_M\}$ be an M-dimensional array whose m-th element is one of the possible outcomes of the m-th measurement $\mathcal{M}_m \in \mathbf{M}$. We denote by $s = 1,\dots,S$ and $m = 1,\dots,M$ discrete indices labelling the elements of $\mathbf{S}$ and $\mathbf{M}$, respectively. The summation over every index in $\mathbf{w}$ but the m-th one, which is set equal to w, is concisely written as follows,

$$\sum_{w_1,\dots,w_{m-1},w_m=w,\dots,w_M} \to \sum_{\mathbf{w},w_m=w} \tag{5}$$

Definition. Given a game $\mathbf{G} = (\mathbf{S},\mathbf{M})$, the set $\mathcal{V}(\mathbf{G})$ contains any conditional probability $\rho(\mathbf{w}|s)$ whose marginal distribution of the m-th variable is the quantum distribution of the outcome w_m given the quantum state s and the measurement m, for any s, m. In other words, the set $\mathcal{V}(\mathbf{G})$ contains any $\rho(\mathbf{w}|s)$ satisfying the constraints

$$\sum_{\mathbf{w},w_m=w} \rho(\mathbf{w}|s) = P_Q(w|s,m), \forall s,m \text{ and } w, \tag{6}$$

where $P_Q(w|s,m) \equiv \langle\psi_s|\hat{E}_{m;w}|\psi_s\rangle$ is the quantum probability of getting the w-th outcome $\hat{E}_{m;w}$ of the measurement $\mathcal{M}_m$ given the quantum state $|\psi_s\rangle$.

The set $\mathcal{V}(\mathbf{G})$ is surely non-empty. A function in $\mathcal{V}(\mathbf{G})$ is $\rho(\mathbf{w}|s) = P_Q(w_1|s,1) \times \dots \times P_Q(w_M|s,M)$, where the variables $w_1,\dots,w_M$ are uncorrelated. The definition of $\mathcal{V}(\mathbf{G})$ can be easily extended to any general probabilistic theory, where $P_Q(w|s,m)$ is replaced by different conditional probabilities. For the sake of concreteness, we will refer to the quantum case, but the following discussion does not rely on any precise form of $P_Q(w|s,m)$ and applies to more general theories.

A pivotal classical protocol for the quantum game $\mathbf{G}$ is as follows.

Master protocol. Alice generates the array $\mathbf{w}$ according to a conditional probability $\rho(\mathbf{w}|s) \in \mathcal{V}(\mathbf{G})$. Then, she sends $\mathbf{w}$ to Bob. Bob simulates the measurement $\mathcal{M}_m$ by generating the outcome w_m.

The definition of $\mathcal{V}(\mathbf{G})$ implies that this protocol exactly simulates the quantum game $\mathbf{G}$. A classical channel from a variable x_1 to x_2 is defined by the conditional probability of getting x_2 given x_1. Its capacity is the maximum of the mutual information between x_1 and x_2 over the space of probability distributions $\rho(x_1)$ [7]. Using the strategy discussed in Ref. [3] and the reverse Shannon theorem [5], it is possible to prove that a master protocol can be turned into a child protocol for parallel simulations whose asymptotic communication cost is equal to the capacity of the classical channel $\rho(\mathbf{w}|s)$.

Lemma 1. Given a conditional probability $\rho(\mathbf{w}|s) \in \mathcal{V}(\mathbf{G})$, there is a child protocol, simulating in parallel R quantum games $\mathbf{G}$, whose asymptotic communication cost per game is equal to the capacity of the channel $\rho(\mathbf{w}|s)$ as R goes to infinity.

Proof. In a simulation of R games $\mathbf{G}$ through R master protocols performed in parallel, Alice sends an array $\mathbf{w}$ to Bob for each game. This array is generated with probability $\rho(\mathbf{w}|s)$. Let $C(\mathbf{W}|S)$ be the capacity of the channel $T: s \to \mathbf{w}$. The child protocol is as follows. Instead of sending $\mathbf{w}$, Alice sends an amount of information, say $\mathcal{C}(R)$, that allows Bob to generate $\mathbf{w}$ for every game $\mathbf{G}$ according to the probability $\rho(\mathbf{w}|s)$. The reverse Shannon theorem states that this can be accomplished with a cost $\mathcal{C}(R)$ such that $\lim_{R\to\infty} \mathcal{C}(R)/R = C(\mathbf{W}|S)$, provided that the receiver and sender share some random variable. □

The first main result is the following theorem about the asymptotic communication complexity. Later on, we will consider the single-shot case.

Theorem 1. The asymptotic communication complexity of the game $\mathbf{G} = (\mathbf{S}, \mathbf{M})$ is the minimum of the capacity of the classical channels $\rho(\mathbf{w}|s)$ in the set $\mathcal{V}(\mathbf{G})$.

Theorem 1 states that the asymptotic communication complexity of the game $\mathbf{G}$ is equal to the quantity

$$\mathcal{D}(\mathbf{G}) \equiv \min_{\rho(\mathbf{w}|s)\in\mathcal{V}(\mathbf{G})} \left(\max_{\rho(s)} I(\mathbf{W}; S) \right), \tag{7}$$

where $I(\mathbf{W}; S)$ is the mutual information between the stochastic variables $\mathbf{w}$ and s.

Proof. Lemma 1 implies that $\mathcal{C}_{min}^{asym}(\mathbf{G}) \le \mathcal{D}(\mathbf{G})$. We show that $\mathcal{C}_{min}^{asym}(\mathbf{G})$ is actually equal to $\mathcal{D}(\mathbf{G})$ by proving that the asymptotic communication cost cannot be smaller than $\mathcal{D}(\mathbf{G})$. Let $\mathcal{C}_0$ be the asymptotic communication cost of a parallel simulation of the game $\mathbf{G}$. We denote by R the number of games $\mathbf{G}$ that are simulated in parallel. In the simulation, Alice sends a variable k generated with conditional probability $\rho(k|y, s^1, \dots, s^R)$, where s^i is an index labelling the quantum state of the i-th game. Bob simulates the measurements $\mathcal{M}_{m^1}, \dots, \mathcal{M}_{m^R}$ by generating the outcomes $w^1, \dots, w^R$ according to a conditional probability $P(w^1, \dots, w^R|k, y, m^1, \dots, m^R)$. Let us denote by $P^i(w^i|k, y, m^1, \dots, m^R)$ the marginal probability of the outcome of the i-th game. We introduce the conditional probabilities

$$P^i(\mathbf{w}^i|k, y) = \prod_{m^i} P^i(w^i_{m_i}|k, y, 1, \dots, m^i, \dots, 1), \tag{8}$$

where $\mathbf{w}^i \equiv \{w^i_1, \dots, w^i_M\}$. Note that we have multiplied over m^i and set the other indices equal to 1. For our purposes, any other choice of the values of the $R-1$ indices would be fine. We use $P^i(\mathbf{w}^i|k, y)$ to build the conditional probability

$$P(\mathbf{w}^1, \dots, \mathbf{w}^R|k, y) = \prod_i P^i(\mathbf{w}^i|k, y). \tag{9}$$

Finally, from this distribution and $\rho(k|y, s^1, \dots, s^R)$, we build the conditional probability

$$\begin{gathered}\rho(\mathbf{w}^1, \dots, \mathbf{w}^R|s^1, \dots, s^R) = \\ \sum_k \int dy \rho(y) P(\mathbf{w}^1, \dots, \mathbf{w}^R|k, y) \rho(k|y, s^1, \dots, s^R).\end{gathered} \tag{10}$$

From the data-processing inequality [7], we have that the capacity, say $C(\mathbf{W}^1, \dots, \mathbf{W}^R|S^1, \dots, S^R)$, of $\rho(\mathbf{w}^1, \dots, \mathbf{w}^R|s^1, \dots, s^R)$ is smaller than or equal to the communication cost $R\,\mathcal{C}_0 + o(R)$, that is,

$$C(\mathbf{W}^1, \dots, \mathbf{W}^R|S^1, \dots, S^R) \le R\,\mathcal{C}_0 + o(R). \tag{11}$$

By construction, we have the constraints

$$\sum_{\mathbf{w}^1,\dots,\mathbf{w}^R, w^i_m = w} \rho(\mathbf{w}^1, \dots, \mathbf{w}^R|s^1, \dots, s^R) = P_Q(w|s_i, m), \tag{12}$$

the left-hand side being the marginal distribution of the variable w^i_m (renamed w) given $s^1, \dots, s^R$. Let $\rho_0(\mathbf{w}|s)$ be the probability distribution in $\mathcal{V}(\mathbf{G})$ with minimal capacity $\mathcal{D}(\mathbf{G})$. The probability distribution

$$\rho_{min}(\mathbf{w}^1, \dots, \mathbf{w}^R|s^1, \dots, s^R) \equiv \prod_i \rho_0(\mathbf{w}^i|s^i), \tag{13}$$

is the channel satisfying constraints (12) with minimal capacity. The minimum is equal to $R\,\mathcal{D}(\mathbf{G})$. Thus,

$$R\,\mathcal{D}(\mathbf{G}) \le C(\mathbf{W}^1, \dots, \mathbf{W}^R|S^1, \dots, S^R). \tag{14}$$

From this inequality and Inequality (11) we have that

$$R\mathcal{D}(\mathbf{G}) \le R\mathcal{C}_0 + o(R). \tag{15}$$

The theorem is proved. □

III. Example 1: Quantum Channel

To illustrate these results, we have evaluated the communication complexity of the following game $\mathbf{G}$. Let us denote by $\vec{v}_x$ the tridimensional vectorial function $\left(\cos\frac{\pi x}{M}, \sin\frac{\pi x}{M}, 0\right)$, where x is a real number. The measurements are projections in a two-dimensional Hilbert space. The eigenvectors of the m-th measurement in $\mathbf{M}$ correspond to the Bloch vectors $\pm\vec{v}_m$ with $m = 1, \dots, M$ and outcomes $w = \pm 1$. The set $\mathbf{S}$ contains all the $2M$ eigenvectors, that is, $\vec{v}_s$ with $s = 1, \dots, 2M$. The quantum probability of getting w given s and m is

$$P_Q(w|s, m) = \frac{1}{2}\left\{1 + w\cos\left[\frac{\pi}{M}(s - m)\right]\right\}. \tag{16}$$

We have evaluated algebraically the asymptotic communication complexity up to $M = 4$. The distributions $\rho(\mathbf{w}|s) \in \mathcal{V}(\mathbf{G})$ with minimal capacity for $M = 2, 3, 4$ are summarized by the analytical equation

$$\rho(\mathbf{w}|s) = \sum_{k=1}^{2M} P(\mathbf{w}|k)\rho(k|s) \tag{17}$$

with

$$\rho(k|s) = \sin\left(\frac{\pi}{2M}\right)\vec{v}_{k+p/2}\cdot\vec{v}_s\,\theta\left(\vec{v}_{k+p/2}\cdot\vec{v}_s\right), \tag{18}$$

$$P(\mathbf{w}|k) = \prod_{m=1}^{M} \theta\left(w_m\vec{v}_m\cdot\vec{v}_{k+p/2}\right), \tag{19}$$

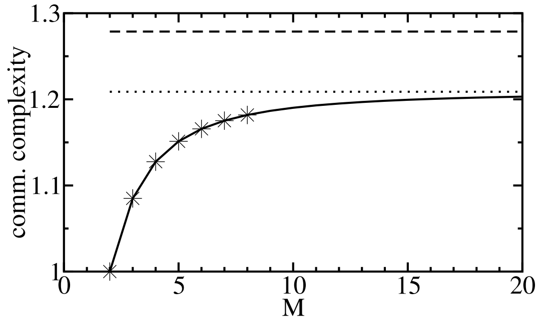

Figure 1. Asymptotic communication complexity for $M = 2, \dots, 20$ measurements (the solid line interpolates the data as a guide of eyes). The dot line represents the asymptotic limit of the function for $M \to \infty$. The dashed line is the communication cost of the model in Ref. [3], working for any projective measurement on the qubit. The stars are the values numerically tested.

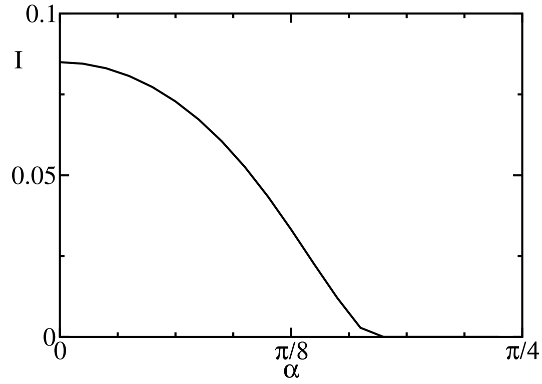

Figure 2. Asymptotic communication complexity for $M = 3$ measurements and different quantum states with $\alpha = 0, \dots \pi/4$. The communication cost for $\alpha < \pi/4$ is not zero in general, but for the bases we have tested the value had the same dimension as the expected numeric error. For $M = 4, 5, 6$ one can see similar, monotonous curves.

where $p = 0$ (1) if M is odd (even) and θ is the Heaviside function. It is easy to prove that $\rho(\mathbf{w}|s)$ is an element of $\mathcal{V}(\mathbf{G})$ for any $M \geq 2$.

Since $P(\mathbf{w}|k)$ is a noiseless channel, the capacity of $\rho(\mathbf{w}|s)$ is equal to the capacity of $\rho(k|s)$. Thus, we find that the asymptotic communication complexity is

$$\mathcal{C}_{min}^{asym}(\mathbf{G}) = \mathcal{N} \sum_{n=\frac{1-M}{2}}^{\frac{M-1}{2}} \cos\left(\frac{\pi n}{M}\right) \log\left[2M\mathcal{N}\cos\left(\frac{\pi n}{M}\right)\right], \tag{20}$$

where $\mathcal{N} = \sin\left(\frac{\pi}{2M}\right)$. Note that the sum index n is not an integer when M is even. We have numerically verified the validity of Eq. (20) for M up to 8 (see Fig. 1). By symmetry, the distribution $\rho(s)$ in Eq. (7) is taken uniform. Thus the minimax problem is reduced to a minimization problem.

If we extrapolate this equation to arbitrary M, we have $\lim_{M\to\infty} \mathcal{C}_{min}^{asym}(\mathbf{G}) = 1 + \log_2 \frac{\pi}{e} \simeq 1.2088$. This value is the asymptotic communication complexity of a noiseless quantum channel with the constraint that the quantum states and the eigenstates of the measurements correspond to Bloch vectors lying on a plane. In Ref. [3], we found a protocol for any quantum state and projective measurements with communication cost equal to $\log_2(4/\sqrt{e}) \simeq 1.2786$, which is about 6% higher. It is likely that this value is actually the asymptotic communication complexity of the quantum channel for general projective measurements.

IV. Example 2: Entanglement Simulation

The introduced method is generic and can be adapted to be used for, e.g., determining the asymptotic communication cost of entanglement simulation instead of quantum channels. For instance, what is the classical-communication rate for simulating singlets? Similarly as before, we define a game $\mathbf{G}$ and measurement bases $\vec{v}_x = \left(\cos\frac{\pi x}{M}, \sin\frac{\pi x}{M}, 0\right)$ for $x = 1, 2, ..., M$, M is the number of bases. The $\vec{v}_x$ are regularly distributed in the XY-plane. We use the same set of vectors for Alice and Bob. Our analysis yields exactly the same results as for the simulation of the quantum channel, reduced by exactly one bit, as expected.

Of special interest is the asymptotic classical communication necessary for simulating pure yet not-maximally entangled qubit pairs. It has been suggested [9] that in the one-shot scenario, there is *no monotonous dependency* of the amount of communication from the entanglement strength. This had been shown with respect to different non-locality measures such as the number of required PR boxes or the violation of some Bell inequality. Our results indicate that this might be different in the *asymptotic* setting.

We use regularly distributed vectors in the YZ plane for the settings. More explicitly, the vectors are $\vec{v}_x' = \left(0, \cos\frac{\pi x}{M}, \sin\frac{\pi x}{M}\right)$ for $x = 1, 2, ..., M$; M is the number of bases. We observe a monotonous connection between entanglement and asymptotic communication complexity.

V. Conclusion

We have shown that results of classical information theory, such as the reverse Shannon channel-coding theorem, can be used for determining the asymptotic classical-communication rate of the behavior, under measurements, of quantum primitives such as quantum channels or entangled systems. Our method is generic and potentially applicable to a wide range of further applications.

We have shown in detail how our procedure allows for evaluating the communication complexity of channels in any general probabilistic theory, in particular quantum theory. It not only relies on the reverse Shannon theorem, but also on and a strategy introduced in Refs. [3], [6]; it is constructive and provides a method for deriving the most efficient protocol that classically simulates a channel. More explicitly, given a quantum channel, we have defined a set $\mathcal{V}$ of classical channels and proved that the minimal classical capacity in $\mathcal{V}$ is the asymptotic communication complexity of the quantum channel. Thus, the problem of evaluating the communication complexity is reduced to a minimax problem. Using the reverse Shannon theorem, the channel in $\mathcal{V}$ with minimal capacity can be turned into the most efficient classical protocol for simulating the quantum channel. We have illustrated this procedure by evaluating the asymptotic communication complexity of a noiseless quantum channel with capacity 1 qubit and some finite sets of quantum states and measurements.

We have discussed that our technique can be adapted to other problems, e.g., calculating the communication complexity of quantum entanglement. In the case of the asymptotic communication complexity of non-maximally entangled qubit pairs, we have found that the results in the asymptotic case may contrast the single-shot scenario [9].

Our procedure is numerically very stable, but the computational time of the minimax routine can grow exponentially with the number of quantum states and measurements. Thus, specific strategies reducing the computational complexity need to be devised in the case of a high number of states and measurements.

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