A Property of the Intrinsic Mutual Information

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Abstract — **The so-called *intrinsic mutual information* is an important measure in the context of information-theoretic secret-key agreement. We prove a property of this information measure which, in particular, strongly simplifies its computation. More generally, our result is useful for analyzing the correlation of two random variables conditioned on a third one.**

I. Definitions and Motivation

The *intrinsic (mutual) information* [2] between two discrete random variables X and Y, given a third random variable Z, is defined as

$$I(X;Y{\downarrow}Z) := \inf_{\overline{Z}} I(X;Y|\overline{Z}) ,$$

where the infimum is taken over all discrete random variables $\overline{Z}$ such that $XY \to Z \to \overline{Z}$ is a Markov chain. This minimization includes, in other words, all discrete conditional probability distributions, or discrete channels, $P_{\overline{Z}|Z}$.

The intrinsic information is useful in a context where two parties, being connected by a public channel, and having access to (repeated realizations of) random variables X and Y, respectively, want to generate a key being secret even if a possible adversary possesses some knowledge, specified by Z. In fact, it was shown [2] that $I(X;Y{\downarrow}Z)$ is an upper bound on the rate $S = S(X;Y||Z)$ at which such a key can be extracted. Another recent result [3] states that $I(X;Y{\downarrow}Z)$ is a lower bound on the rate at which secret-key bits are required for distributing pieces of information X and Y by public communication, leaving a possible wire-tapper with no more information than Z.

Since the intrinsic information is defined by an infimum ranging over the set of all possible discrete conditional probability distributions $P_{\overline{Z}|Z}$, it is a priori not easy to compute. In particular, to prove that $I(X;Y{\downarrow}Z) > 0$ holds, it is not enough to show that $I(X;Y|\overline{Z})$ is strictly positive for all Markov chains $XY \to Z \to \overline{Z}$: The minimum might not be attained by any particular channel since the space of discrete channels is not a compact set. Our result is a step towards the better understanding of $I(X;Y{\downarrow}Z)$: We prove that the minimum *is indeed taken* by a specific channel $P_{\overline{Z}|Z}$ and, moreover, that this minimum can be reached for a channel whose output alphabet is not larger than the alphabet of Z.

As a consequence, the following is true for all random variables X, Y, and Z (where the range $\mathcal{Z}$ of Z is finite): If there exists a Markov chain $XY \to Z \to \overline{Z}$ such that $I(X;Y|\overline{Z}) = 0$ holds, then there exists a Markov chain $XY \to Z \to \overline{Z}_{\text{fin}}$, where $\overline{Z}_{\text{fin}}$ is now a *finite* random variable with range $\overline{\mathcal{Z}}_{\text{fin}} = \mathcal{Z}$, such that $I(X;Y|\overline{Z}_{\text{fin}}) = 0$ holds.

II. Main Results and Conclusions

Theorem. *If the range $\mathcal{Z}$ of Z is finite, then there exists a finite random variable $\overline{Z}$, having the same range $\mathcal{Z}$, such that $XY \to Z \to \overline{Z}$ is a Markov chain and*

$$I(X;Y \downarrow Z) = I(X;Y|\overline{Z}) .$$

The infimum over discrete channels from Z to $\overline{Z}$ in the definition of the intrinsic information can thus be replaced by a minimum over channels with output alphabet $\mathcal{Z}$.

Corollary 1. *If the range $\mathcal{Z}$ of Z is finite, then*

$$I(X;Y \downarrow Z) = \min_{\overline{Z}} I(X;Y|\overline{Z})$$

where the minimum is taken over all random variables $\overline{Z}$ with range $\mathcal{Z}$ such that $XY \to Z \to \overline{Z}$ is a Markov chain.

In particular, this result simplifies the task of proving that the intrinsic information of a given triple of random variables is non-vanishing [1].

If and only if $I(X;Y|\overline{Z})$, the mutual information of random variables X and Y with respect to $\overline{Z}$, vanishes, then X and Y are independent conditioned on $\overline{Z}$. This immediately proves the following corollary.

Corollary 2. *If the range $\mathcal{Z}$ of Z is finite, then the following statements are equivalent:*

1. *There exists a discrete random variable $\overline{Z}$ such that $XY \to Z \to \overline{Z}$ is a Markov chain, and X and Y are independent conditioned on $\overline{Z}$.*
2. *There exists a finite random variable $\overline{Z}$ with range $\mathcal{Z}$ such that $XY \to Z \to \overline{Z}$ is a Markov chain, and X and Y are independent conditioned on $\overline{Z}$.*
3. *$I(X;Y \downarrow Z) = 0$.*

References

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