Violation of Leggett-Garg Inequalities Implies Information Erasure

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***Abstract*—The Leggett-Garg inequalities were originally introduced for experimentally testing a possible break of the quantum evolution in mesoscopic systems. In this paper, we take a different point of view by focusing on faithful classical simulations of sequential quantum measurements. In this context, the violation of Leggett-Garg inequalities implies that classically simulated quantum measurements induce perturbations into the subsequent evolution of the classical variables. We show that the implication is even stronger and a measurement erases previous information by performing a partial reset on the classical state. Thus, the measuring device acts as a low-temperature bath absorbing entropy from the measured system. Information erasure is a form of preparation contextuality. Our proof is straightforward if one assumes that maximal ignorance of the quantum state is compatible with maximal ignorance of the classical state. We also employ a weaker hypothesis.**

I. INTRODUCTION

The Leggett-Garg (LG) inequalities were originally introduced in a quantum-foundational context for experimentally testing the emergence of macroscopic realism in mesoscopic systems, such as superconducting qubits [1]. First, Leggett and Garg formulated two principles that should hold in the macroscopic world and may be justified by models of wave-function collapse. The two assumptions are called by Leggett and Garg (A1) *macroscopic realism* and (A2) *noninvasive measurability*. Then, from these assumptions, Leggett and Garg derived inequalities which are violated by quantum systems undergoing a unitary evolution between measurements. The experimental violation of the inequalities would be a proof of the failure of one of the two assumptions.

In this paper, we focus on the information-theoretic problem of classically simulating sequential quantum measurements. From this point of view, we take for granted assumption (A1) and the unitary evolution between measurements. Our purpose is to show that the violation of LG inequalities implies more than a mere break of assumption (A2). The interaction with a measuring device can be invasive by increasing, preserving, or decreasing the entropy of the classical state, provided that the measurement outcome is forgotten. In the latter case, we say that the measurement *erases information*. Since the von Neumann entropy never decreases after a projective measurement (provided that the outcome is ignored), one may expect that this feature is inherited by a faithful classical causal simulation of a sequence of quantum measurements. However, we prove that the perturbation induced by a measurement cannot be reproduced classically without a partial reset of the classical state of the measured system. Classically, the measuring device acts as a low-temperature bath absorbing entropy from the system. *Information erasure* comes from the quantum violation of the LG inequalities and an additional hypothesis. The proof is straightforward if we assume that maximal ignorance of the quantum state is compatible with maximal ignorance of the underlying classical state. A weaker hypothesis is the assumption that the entropy of the system is bounded in the simulation. The reset of the quantum state after a measurement is mirrored by the partial reset of the classical state in a classical causal simulation. In Ref. [2], it was shown that the erasure of just one bit suffices to account for the outcome statistics of a two-state system, the measurements being performed at two arbitrary times. Interestingly, the de Broglie-Bohm theory does not satisfy the additional hypothesis of finiteness of the entropy and, indeed, measurements change the state, but do not erase information. The evasion of our conclusion on information erasure recalls the Maxwell demon with infinite memory, who can perform arbitrary operations without erasing its internal state.

The paper is organized as follows. In Sec. II, we introduce the LG inequalities and provide a link to the Bell-like Clauser–Horne–Shimony–Holt (CHSH) [3] inequality. This link suggests our main result on information erasure. In Sec. III, we introduce the classical model simulating sequences of projective measurements. In Sec. IV, the violation of LG inequalities is rephrased in the framework of the classical model. In Sec. V, we prove the theorem on *information erasure*, which is physically interpreted as entropy flow from the system to the low-entropy measuring device. We conclude the section by discussing the relation between information erasure and preparation contextuality [4].

II. LEGGETT-GARG INEQUALITY

In this section, we introduce Leggett-Garg (LG) inequalities and discuss their relationship with the Bell-like CHSH inequality satisfied by local correlations. The LG inequalities refer to a scenario in which a projective measurement $\hat{A}$ is executed at two times t_k and t_l chosen among a set of n values, say

$t_1, \dots, t_n$. A measurement at time t_k gives some value $a_k = \pm 1$. Ruling out retrocausality, assumptions (A1-A2) imply that there is a joint probability distribution $\rho(a_1, \dots, a_n)$ of the values $a_1, \dots, a_n$ independently of the actual execution of the measurements. If assumption (A2) is dropped, the joint probability takes the more general form $\rho(a_1, \dots, a_n | s_1, \dots, s_n)$, where s_k is a binary variable encoding the information on the actual execution of the k-th measurement. If the measurement is executed, then s_k is set equal to 1, otherwise s_k is set equal to 0. The two assumptions can be replaced by the condition

$$\rho(a_1, a_2, \dots | s_1, s_2, \dots) = \rho(a_1, a_2, \dots), \tag{1}$$

which explicitly rules out also retrocausal influence. This equation implies the inequality [1]

$$C_{1,2} + C_{2,3} + C_{3,4} + C_{n-1,n} - C_{1,n} \le n - 2 \tag{2}$$

where $C_{i,j}$ are the correlation functions $\langle a_i a_j \rangle$.

For $n = 4$, the LG inequality takes the form of CHSH inequality [3]. This similarity is not a coincidence and it becomes clearer if we swap t_2 and t_3. We have

$$C_{1,3} + C_{2,3} + C_{2,4} - C_{1,4} \le 2. \tag{3}$$

There are two sets of times $\{t_1, t_2\}$ and $\{t_3, t_4\}$ such that the correlations in the inequality are between an element in one set and an element in the other. We can reframe the narrative as follows. There are two parties, Alice and Bob. Alice chooses to perform a measurement at time t_1 or t_2. Subsequently, Bob chooses to perform a measurement at time t_3 or t_4. In this form, the procedure is conceptually identical to the CHSH scenario in which Alice and Bob choose one of two different measurements. However, the CHSH inequality is derived from the slightly different assumption of locality, which can be stated as follows. The outcomes of Alice and Bob only depend on their own choice and some global random variable, say λ, which is uncorrelated with $s_1, \dots, s_4$. In the LG scenario, the assumption leads to the equation

$$\begin{aligned} &\rho(a_1, \dots, a_4 | s_1, \dots, s_4) = \\ &\int d\lambda \rho(a_1, a_2 | s_1, s_2, \lambda) \rho(a_3, a_4 | s_3, s_4, \lambda) \rho(\lambda). \end{aligned} \tag{4}$$

In general, the distribution $\rho(a_1, \dots, a_4 | s_1, \dots, s_4)$ does not satisfy condition (1), as the assumption of locality does not forbid an influence from t_1 to t_2 and from t_3 to t_4. But this "local" invasivity is irrelevant since Ineq. (3) does not contain the correlations $C_{1,2}$ and $C_{3,4}$. Indeed, the conditional distribution defined by Eq. (4) is operationally equivalent to an unconditional distribution. Let us see this explicitly. Since Bob exclusively executes one measurement at time t_3 or t_4, we always have $s_3 \ne s_4$. On Alice's side, we also have $s_1 \ne s_2$. Thus, we can replace the probability distributions $\rho(a_1, a_2 | s_1, s_2, \lambda)$ and $\rho(a_3, a_4 | s_3, s_4, \lambda)$ with

$$\sum_{a_1', a_2'} \rho(a_1, a_2' | 1, 0, \lambda) \rho(a_1', a_2 | 0, 1, \lambda) \equiv \rho(a_1, a_2 | \lambda)$$

and

$$\sum_{a_3', a_4'} \rho(a_3, a_4' | 1, 0, \lambda) \rho(a_3', a_4 | 0, 1, \lambda) \equiv \rho(a_3, a_4 | \lambda),$$

respectively. These replacements do not change the observed correlations. Thus, the probability distribution of the four outcomes

$$\int d\lambda \rho(a_3, a_4 | \lambda) \rho(a_1, a_2 | \lambda) \rho(\lambda) \equiv \rho(a_1, a_2, a_3, a_4) \tag{5}$$

gives the same correlations as the original distribution $\rho(a_1, \dots, a_4 | s_1, \dots, s_4)$, Hence, also the latter satisfies inequality (3).

The analogy with the CHSH inequality shows an interesting consequence of the violation of the LG inequalities. Namely, a classical causal simulation of sequential measurements must employ some information flow from the past to the future even if the initial quantum state has maximal entropy and signaling is not possible. We will come back to that in Sec. IV.

A. Quantum violation of the LG inequalities

Let us show that quantum theory violates the LG inequality with four times $t_1, \dots, t_4$. The unitary evolution is taken time-independent with Hamiltonian equal to

$$\hat{H} = |1\rangle\langle -1| + |-1\rangle\langle 1|, \tag{6}$$

so that the correlation between a_k and a_l at times t_k and t_l is

$$\langle a_k a_l \rangle = \cos[2(t_k - t_l)]. \tag{7}$$

The left-hand side of the LG inequality is maximal at $t_{k+1} = \pi/8 + t_k$ with $k \in \{1, 2, 3\}$. The maximum is the *Tsirelson bound* $2\sqrt{2}$, which violates the locality bound 2.

We said that some information about Alice's choice needs to be communicated to Bob if the LG are violated. Is this information necessary for every value of t_1 and t_2 on Alice's side? To answer this question, let us find the values of t_1 and t_2 such that the inequality is violated for some t_3 and t_4. Maximizing the left-hand side of the LG inequality with respect to t_3 and t_4, we get the value

$$2\left(|\cos(t_2 - t_1)| + |\sin(t_2 - t_1)|\right) \tag{8}$$

which always violates the LG inequality, apart from the values $t_2 = t_1 + m_1 \pi/2$, m_1 being an integer. These values correspond to the case in which the measurements at time t_1 and t_2 project on the same basis. Thus, whenever the two measurements do not commute, there must be some finite amount of communication from Alice to Bob.

For later convenience, we summarize the previous statements in the following.

Lemma 1: Let us consider the LG scenario with 4 times $t_1, \dots, t_4$. The evolution is generated by Hamiltonian (6). If the unitary transformation from t_1 and t_2 does not preserve the set $\{|1\rangle, |-1\rangle\}$, then there are two values of t_3 and t_4 such that the LG inequality (3) is violated.

III. Classical causal simulation

In this section, we introduce a classical model simulating a projective measurement $\hat{A}$. More precisely, the model simulates a process of sequential quantum measurements $\hat{A}$ on a unitarily evolving quantum system. We call this process a *quantum protocol*.

Definition 1: (*Schrödinger picture*) A quantum protocol is defined by a Hermitian operator $\hat{A}$ of rank n, a sequence of execution times $t_1 < t_2 < t_3 < \dots$, a rank-n density operator $\hat{\rho}$ at some initial time $t_0 \le t_1$ and by the unitary evolutions from time t_k to t_{k+1} with $k \in \{0, 1, 2, 3, \dots\}$. The projective measurement $\hat{A}$ is executed at times $t_1, t_2, \dots$, so that a sequence of measurement-outcomes $a_1, a_2, \dots$ is generated.

There is some excess of irrelevant information in this definition. Instead of specifying the execution times and the unitary evolutions, we can just specify a sequence of measurements in the Heisenberg picture.

Definition 2: (*Heisenberg picture*) A *quantum protocol* is defined by a sequence of projective measurements $\hat{A}_1, \hat{A}_2, \dots$ of rank n and an initial rank-n density operator $\hat{\rho}$. The measurements generate a sequence of outcomes $a_1, a_2, \dots$.

The causal simulation of a quantum protocol is a classical rephrasing of each step of the process. The state of the system at each time is encoded by some element λ in a space Λ. Employing popular terms in quantum foundation, we call λ and Λ an *ontic* state and its ontological space, respectively. The space Λ must be uncountably infinite in a Markov simulation [5], [6], we have to define a measure on it such that unitary evolutions are associated to bijective volume-preserving evolutions in the ontological space. Thus, the differential entropy

$$H(\rho) \equiv -\int d\lambda \rho(\lambda) \log \rho(\lambda) \tag{9}$$

defined on this measure is constant under unitary transformations of the quantum state. We also assume that the ontological space has *finite* volume, so that the entropy is bounded from above. For example, the space may be a compact subset of an Euclidean space or a hypersphere. Our main objective is to show that measurements reduce this entropy under some suitable hypothesis. We define the following.

Model 1: Classical causal simulation of the quantum projective measurement $\hat{A}$:

- At each time, the classical state of the system is some element λ of a measurable *ontological space* Λ with a finite measure (volume).
- A unitary evolution is simulated by a reversible volume-preserving transformation in Λ.
- The execution of the measurement $\hat{A}$ modifies an ingoing value $\lambda^{in} \in \Lambda$ to a outgoing value λ^{out} according to a conditional probability $p_M(\lambda^{out}|\lambda^{in})$. The measurement outcome a is generated with a conditional probability $\rho(a|\lambda^{in})$.
- The value of λ is statistically independent of the execution of future measurements and unitary evolutions (causality).

Model 1 defines building blocks for simulating unitary evolutions and measurements. These blocks can be freely chosen to build a *simulation protocol* of any sequence of measurements on a unitarily evolving quantum system.

The model provides a *faithful* simulation of a projective measurement $\hat{A}$ if every quantum state is associated to a probability distribution $\rho(\lambda)$ such that the outcomes of every sequential set of subsequent quantum measurements of $\hat{A}$ are exactly reproduced by the corresponding simulation protocol.

Given our model, let us consider the LG scenario, in which the actual execution of a measurement at time t_k is controllable and encoded in the binary variable s_k. The value of s_k is 1 or 0 if the measurement has been performed or not, respectively. Let λ_k be the classical state just before time t_k. The causal relation between λ_k and λ_{k+1} is described by a conditional probability of the form

$$P_{\hat{A}_k}(\lambda_{k+1}|\lambda_k, s_k) \equiv \begin{cases} \delta(\lambda_k - \lambda_{k+1}) & s_k = 0 \\ \rho_{\hat{A}_k}(\lambda_{k+1}|\lambda_k) & s_k = 1 \end{cases} \tag{10}$$

where $\hat{A}_k$ is the k-th measurement operator in the Heisenberg picture. Ignoring the outcomes of the measurements, the overall process with n times is described by the Markov chain

$$\begin{gathered} \rho(\lambda_1, \dots, \lambda_{n+1}|s_1, \dots, s_n) \equiv \\ P_{\hat{A}_n}(\lambda_{n+1}|\lambda_n, s_n) \dots P_{\hat{A}_1}(\lambda_2|\lambda_1, s_1)\rho(\lambda_1), \end{gathered} \tag{11}$$

where $\rho(\lambda_1)$ is the probability distribution of the ontic state before the first measurement. Employing causality, this distribution does not depend on $s_1, \dots, s_n$.

IV. Rephrasing LG inequalities

In this section, we prove that, given two non-commuting measurements $\hat{A}_1$ and $\hat{A}_2$, the outgoing distribution $\rho(\lambda)$ after $\hat{A}_1$ is *always* different from the distribution emerging after $\hat{A}_2$, regardless of the quantum states before the measurements. In other words, the state λ always contains information on which measurement is actually executed. This property captures the very essence of the quantum violation of LG inequalities in the framework of Model 1.

Let $\rho_1(\lambda)$ and $\rho_2(\lambda)$ be the ingoing distributions before measurement $\hat{A}_1$ and $\hat{A}_2$, respectively. The transition probabilities $\rho_{\hat{A}_1}(\lambda|\lambda^{in})$ and $\rho_{\hat{A}_2}(\lambda|\lambda^{in})$ are associated to the two measurements.

Definition 3: Given the two measurements $\hat{A}_1$ and $\hat{A}_2$, a classical channel $M[\hat{A}_1, \hat{A}_2] : R \to \Lambda$ from a bit $r \in \{1, 2\} \equiv R$ to an ontic state $\lambda \in \Lambda$ is defined by the conditional distribution

$$\rho(\lambda|r) = \int d\lambda^{in} \rho_{\hat{A}_r}(\lambda|\lambda^{in})\rho_r(\lambda^{in}). \tag{12}$$

In general, there are infinite channels M associated to a pair of measurements, as they may depend on the probability distributions prior to the measurement.

The violation of LG inequality (3) implies the following.

Theorem 1: Given two incompatible measurements $\hat{A}_1$ and $\hat{A}_2$ in one-qubit Model 1, the conditional probability $\rho(\lambda|r)$ of the channel $M[\hat{A}_1, \hat{A}_2]$ is different from $\rho(\lambda)$.

Proof. As implied by Lemma 1, there are two other measurements $\hat{A}_3$ and and $\hat{A}_4$ such that the LG inequality (3) is violated for the sequence of measurements $\hat{A}_1, \dots, \hat{A}_4$. Two of the four measurements are executed in each instance. One measurement is chosen in the set $S_A = \{\hat{A}_1, \hat{A}_2\}$, the other in $S_B \in \{\hat{A}_3, \hat{A}_4\}$. Let us denote by a_A and b_B the outcome of the measurement in the set S_A and S_B, respectively. The probability distribution of the outcomes has the form

$$\begin{array}{c} \rho(a_A, a_B|r_A, r_B) \equiv \\ \int d\lambda d\chi \rho(a_B|\lambda, r_B)\rho(a_A|\chi, r_A)\rho(\lambda|\chi, r_A)\rho(\chi), \end{array} \tag{13}$$

where $r_A, r_B \in \{1, 2\}$ and the executed measurements are $\hat{A}_{r_A}$ and $\hat{A}_{r_B+2}$. The variable χ and its distribution $\rho(\chi)$ summarize the information about the ontic state prior the first measurement. Let us assume that the statement is false, then

$$\rho(\lambda|r_A)|_{r_A=1} = \rho(\lambda|r_A)|_{r_A=2} .$$

By Bayes' theorem, the distribution $\rho(a_A, a_B|r_A, r_B)$ takes the form

$$\begin{array}{c} \rho(a_A, a_B|r_A, r_B) \equiv \\ \int d\lambda d\chi \rho(a_B|\lambda, r_B)\rho(a_A|\lambda, \chi, r_A)\rho(\lambda|r_A)\rho(\chi). \end{array} \tag{14}$$

Since $\rho(\lambda|r_A) = \rho(\lambda)$, we have

$$\begin{array}{c} \rho(a_A, a_B|r_A, r_B) \equiv \\ \int d\bar{\chi} \rho(a_B|\bar{\chi}, r_B)\rho(a_A|\bar{\chi}, r_A)\rho(\bar{\chi}) \end{array} \tag{15}$$

for some shared random variable $\bar{\chi}$. As discussed in Sec. II, this distribution satisfies the LG inequality, in contradiction with the premise. □

This theorem captures the very essence of the quantum violation of LG inequalities. It is plausible that the theorem holds for any dimension of the Hilbert space, but we consider only the case of single qubits.

V. Information erasure

In Sec. II we remarked that the analogy between Leggett-Garg and CHSH inequalities suggests something interesting about the kind of invasivity a quantum measurement exerts on the measured system. An interaction can modify a system by increasing, preserving, or decreasing the entropy. The latter case is conventionally called *information erasure*, in the sense that the system undergoes some degree of state reset. The violation of the LG inequalities can be reproduced by the classical model 1 only if there is a communication of the choices of $s_1, \dots, s_n$ flowing from the past to the future. This communication is implied by Theorem 1. However, communication is possible only if the carrier of the information has initially a low entropy or its state can be erased by a low-entropy external device. Suppose that the initial quantum state has maximal von Neumann entropy and that this corresponds to maximal ignorance of the classical state. From the violation of the LG inequalities, the execution of a measurement must be encoded in the classical state λ of the system, which is the only carrier of information in the model. Since the system has initially maximal entropy, the measuring device has to exert an *information erasure* on the system.

Before proving the *information-erasure* theorem, let us give some definitions. Let $\rho_{uni}(\lambda)$ be the probability distribution of the ontic state which is uniform with respect to the space measure. We say that ρ_{uni} represents the state of *maximal ignorance* of the ontic state. Information erasure can be proved by assuming the following.

Assumption 1: A quantum state with maximal von Neumann entropy, say $\hat{\rho}_{max}$, is compatible with maximal ignorance of the ontic state.

The state of maximal ignorance does not need to be 'physically' attainable in the model as a convex composition of distributions obtained by sequence of measurements. It is sufficient that the uniform distribution exists and generates faithfully the statistics of the quantum state $\hat{\rho}_{max}$.

Definition 4: A measurement $\hat{A}$ erases information in Model 1 if the entropy of

$$\rho(\lambda) \equiv \int d\lambda^{in} \rho_{\hat{A}}(\lambda|\lambda^{in})\rho_0(\lambda^{in}) \tag{16}$$

is lower than the entropy of some distribution $\rho_0(\lambda^{in})$ compatible with a quantum state.

Theorem 2: If Assumption 1 holds in one-qubit Model 1, then measurements erase information.

Proof. Let us prove that a measurement $\hat{A}_1$ erases information in Model 1. Let the initial probability distribution be $\rho_{uni}(\lambda)$. A second measurement $\hat{A}_2$ is defined by some unitary evolution and subsequent measurement of $\hat{A}_1$ such that $\hat{A}_1$ and $\hat{A}_2$ are incompatible. Let us assume that measurement $\hat{A}_1$ does not erase information. Thus, the outgoing probability distribution has maximal entropy, that is, it is equal to $\rho_{uni}(\lambda)$. Since unitary evolutions preserve the distribution ρ_{uni}, also measurement $\hat{A}_2$ has outgoing distribution ρ_{uni}, in contradiction with Theorem 1. Thus, $\hat{A}_1$ erases information. □

Provided that Theorem 1 holds in any dimension of the Hilbert space, the information-erasure theorem can be trivially extended to the case of many qubits. In Ref. [2], it was shown that the erasure of just one bit suffices to account for the outcome statistics of a two-state system, the measurements being performed at two arbitrary times.

A. Weakening Assumption 1

Assumption 1 is necessary to infer information erasure of measurements. Indeed, the ontic state λ cannot contain information about the execution of previous measurements without erasure if the initial ontic state is completely unknown. A completely unknown initial state λ acts like the cryptograpic key in the one-time-pad algorithm. Now, suppose that every probability distribution $\rho(\lambda)$ associated with $\hat{\rho}_{max}$ does not have maximal entropy. For example, we could have a peaked distribution ρ_0 on the ontological space Λ. After the measurement, this distribution can be shifted without decreasing its entropy. The resulting distribution ρ_1 is distinct from the initial distribution, so that it contains information about the actual execution of the measurement. Thus, in principle, measurements without information erasure are compatible with quantum theory. However, note that the statistical mixture of the distributions ρ_0 and ρ_1 has higher entropy with respect to the initial distribution ρ_0. Imagine the scenario in which a measurement $\hat{A}$ is performed with some probability, say $1/2$, at time t and the measurement does not erase information.

The entropy of the ontic state after the measurement generally increases. Sequentially executing this procedure with incompatible measurements, the overall increase of the entropy can be arbitrarily large. Suppose now that the quantum state $\hat{\rho}$ is compatible with a probability distribution $\rho(\lambda)$ with finite entropy, that is, the entropy is not $-\infty$. After a finite number of measurements, the entropy saturates to a maximum value, after which information erasure is necessary to simulate the quantum statistics of subsequent measurements. Thus, let us replace Assumption 1 with the following.

Assumption 2: There is a quantum state $\hat{\rho}$ compatible with a distribution $\rho(\lambda)$ whose entropy is finite.

Theorem 3: If Assumption 2 holds in one-qubit Model 1, then measurements erase information.

Proof. Let us assume that measurement $\hat{A}$ does not erase information. We execute the measurement at each time of a sequence $t_1, t_2, \dots$ with probability $1/2$. The unitary evolution between two consecutive times t_k and t_{k+1} is taken independent of k. In the Heisenberg picture, the measurements are associated to the operators $\hat{A}_1, \hat{A}_2, \dots$. There is a unitary evolution such that two consecutive measurements $\hat{A}_k$ and $\hat{A}_{k+1}$ are incompatible for every $k \geq 1$. By Assumption 2, the initial distribution has finite entropy, say S_{min}. As the unitary evolution and the measurements cannot decrease the entropy, the entropy is finite at every time. Let us show that the entropy increases from time t_k to time t_{k+1} for every $k \geq 1$. Let $\rho_0(\lambda)$ be the probability distribution just before time t_k. We have 4 cases occurring each with probability $1/4$: (1) No measurement is executed at times t_k and t_{k+1}. (2) Both the measurements are executedr. (3) Only the measurement $\hat{A}_k$ is executed. (4) Only the measurement $\hat{A}_{k+1}$ is executed. The four cases are associated to 4 outgoing distributions, say $\rho_1, \dots, \rho_4$, just after time t_{k+1}. Since the measurements do not decrease the entropy, the probability distribution $1/4 \sum_{k=1}^{4} \rho_k$ has the same entropy of ρ_0 only if $\rho_k = \rho_l$ for $k, l \in \{1, 2, 3, 4\}$. This is in contradiction with Theorem 1. Thus, the entropy just after time t_{k+1} is greater than the entropy just before t_k. The entropy difference is lower-bounded by some constant $\sigma > 0$ for every pair. Thus, the entropy increases at least linearly along the sequence. But this is not possible because the entropy is upper bounded (first condition in Model 1). Thus, the measurement $\hat{A}$ erases information. □

It is possible to prove that the quantum state $\hat{\rho}_{max}$ with maximal entropy is associated with a probability distribution $\rho_{max}(\lambda)$ whose entropy is decreased by the measurement.

B. Interpretation of information erasure

Information erasure can have a simple justification once we consider the overall process behind a measurement. No measurement is possible if some external system with lower entropy is not available. A measurement device can be modeled as a pointer at some rest position and getting entangled with the measured system after an interaction. This modeling of a quantum measurement does not work if the initial state of the pointer is completely unknown. Thus, the device can be seen as a kind of 'low temperature' bath that 'cools' the system during the measurement with a transfer of entropy from the latter to the former.

C. Preparation contextuality

In Ref. [4], Spekkens showed that any ontological rephrasing of quantum theory is *preparation contextual*. Namely, there are mixed quantum states whose associated probability distribution $\rho(\lambda)$ on the ontological space depend on the preparation context. For example, there are infinite ways for representing a maximally mixed state $\hat{\rho}_{max}$ as convex combination of pure states. In a non-contextual ontological theory, these different representations should correspond to the same distribution $\rho(\lambda)$. This turns out to be false. Indeed, information erasure is an example of preparation contextuality. Suppose that a qubit is in the maximally mixed state $\hat{\rho}_{max}$. There is a probability distribution $\rho_{max}(\lambda)$ associated to $\hat{\rho}_{max}$ such that a measurement $\hat{A}$ transforms $\rho_{max}(\lambda)$ to a different distribution with lower entropy. Since we trace out the outcome, the quantum state after the measurement is still $\hat{\rho}_{max}$. Thus, we have two preparation procedures which are operationally identical, but generate different distributions. In one procedure, we take a maximally mixed quantum state and we do nothing else. In the second procedure, we take the maximally mixed state, we execute the measurement $\hat{A}$ and forget the result.

VI. Conclusions

Considering classical simulations of multiple projective measurements, we have shown that the interaction of a system with a measuring device erases classical information carried by the simulated system. This can be interpreted as a flow of entropy from the system to the device. The proof needs an assumption of finiteness of the entropy. *Information erasure* emerging in classical simulations is not displayed at the level of the quantum formalism, in which projective measurements never reduce the Neumann entropy if their outcome is ignored.

In perspective, it is useful to quantify the minimal amount of information that a measurement must erase in a classical sinulation. These further studies have relevance in quantum communication complexity and, potentially, in quantum cryptography. For example, information erasure implies that there are scenarios in which quantum channels offer an advantage over classical channels. Finally, information erasure comes from the assumption of causality. We leave open the extension of these results to more general non-causal models.

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