Towards Characterizing the Non-Locality of Entangled Quantum States

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Abstract — **We propose a so-called pseudo-telepathy game for n players demonstrating the non-locality of quantum information. The simplicity of its classical analysis contrasts the often quite involved analysis of previously proposed such games [1]. Moreover, our game allows for a quantitative characterization of entanglement in terms of communication complexity.**

I. The Game

Consider the following game $\mathcal{G}_n^s$ involving $n \geq 5$ collaborating players $P_1, \ldots, P_n$. First, two players P_i and P_j are chosen randomly in such a way that neither of them knows who the other one is. (The non-chosen, remaining, $n-2$ players can be allowed to know which pair of players was chosen.)

The remaining players are now allowed to communicate in order to generate one "hint" bit b which they say out loud (in particular, P_i and P_j can hear the bit). The chosen players P_i and P_j must then independently (i.e., no communication between them is allowed) generate a bit b_i and b_j, respectively. The game is won simply if $b_i \neq b_j$.

We will show that this game can be won with probability at most (roughly) 75% classically (if n is large enough), but with probability 1 (for any value of n) if the players can share quantum information.

II. Classical Analysis

Let us first consider a classical setting where the players rely on arbitrary *classical* (but no quantum) information which they might have agreed on before the start of the game.

We can assume without loss of generality that each player's strategy is deterministic (since his randomness can assumed to be fixed before the start of the game). Once a player is chosen, the only information he gets is the hint bit b. For any given (deterministic) strategy, this bit b thus completely determines his output. Obviously, there exist exactly *four* possible strategies, namely to output 0, 1, b, or $\overline{b}$ (where $\overline{b}$ denotes the complement of b).

If the strategies of the chosen players are the same, they will clearly output the same bit and the game is lost. (Otherwise, if their strategies are different and if the remaining players know these strategies, they can always win.) Finding the probability of losing the game thus amounts to determining the probability $p(n)$ of the event that two players with the *same* strategy (where four strategies are possible) are picked out of the set of n players. The probability of this event can easily be shown to be $p(5) = 1/10$ for $n = 5$ players, and $p(n) \to 1/4$ for $n \to \infty$.

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III. A Winning Strategy for Quantum Players

If the players can not only share classical information, but are additionally allowed to store a quantum state being generated and shared before the game starts, they can win the game *with certainty.* (Note that, during the game, the players are only allowed to process the quantum information locally, i.e., an external observer would not be able to detect that the players follow a quantum strategy.)

Assume that each of the n players initially controls one qubit of a n-partite GHZ state. Then, the winning quantum strategy is as follows: After the two players P_i and P_j have been chosen, the remaining players first measure their qubits with respect to the diagonal basis, and then announce the parity of their measurement outcomes to the chosen players as the hint bit b. Each of the two chosen players then determines his output bit by measuring his subsystem, depending on this hint bit b, in either the diagonal or the circular basis.

A straightforward calculation (which is omitted in this extended abstract) shows that, with this strategy, the probability of the event that the two chosen players both have the same measurement outcome is zero, i.e., they will always have different outputs and thus win the game.

IV. Quantitative Results

Whereas, for quantum players, communicating only one hint bit to the chosen players suffices to win the game $\mathcal{G}_n^s$, the number of bits $C_{\rm win}^{\rm cl}(\mathcal{G}_n^s)$ to be communicated by classical players in order to win $\mathcal{G}_n^s$ can be shown to be bounded by $C_{\rm win}^{\rm cl}(\mathcal{G}_n^s) \geq \log_2 \log_2 n$. This immediately gives a bound for the amount of communication necessary to classically simulate (cf. [2]) the classical outcomes of local measurements on two (arbitrarily chosen) 2-dimensional subsystems being in a GHZ state with a $(n-2)$-qubit subsystem.

An even stronger result is obtained when a slight generalization $\mathcal{G}_n^g$ of the game $\mathcal{G}_n^s$ is considered, where not only 2, but an arbitrary number k of the n players are arbitrarily chosen. It can be shown that in order to win $\mathcal{G}_n^g$, at least $C_{\rm win}^{\rm cl}(\mathcal{G}_n^g) \geq \frac{1}{2}\log_2 n - 2$ bits have to be communicated by classical players, whereas, for quantum players sharing a GHZ state, one bit still suffices.

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References

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