An All-Or-Nothing Flavor to the Church-Turing Hypothesis

Stefan Wolf

Faculty of Informatics, Università della Svizzera italiana (USI), CH-6900 Lugano.
Facoltà indipendente di Gandria, Lunga scala, CH-6978 Gandria.

Abstract. Landauer's principle claims that "Information is Physical." It is not surprising that its conceptual *antithesis*, Wheeler's "It from Bit," has been more popular among computer scientists — in the form of the *Church-Turing hypothesis:* All natural processes can be computed by a universal Turing machine; physical laws then become descriptions of subsets of *observable*, as opposed to merely *possible*, computations. Switching back and forth between the two traditional styles of thought, motivated by quantum-physical Bell correlations and the doubts they raise about fundamental space-time causality, we look for an intrinsic, physical randomness notion and find one around the second law of thermodynamics. Bell correlations combined with *complexity as randomness* tell us that beyond-Turing computations are either physically impossible, or they can be carried out by "devices" as simple as individual photons.

1 Introduction

1.1 Ice *versus* Fire

According to *Jeanne Hersch* [23], the entire history of philosophy is coined by an antagonism rooting in the fundamentally opposite world views of the pre-Socratic philosophers *Parmenides of Elea* (515 B.C.E. – 445 B.C.E.) on the one hand and *Heraclitus* (520 B.C.E. – 460 B.C.E.) on the other. For Parmenides, any change, even time itself, is simply an *illusion*, whilst for Heraclitus, *change* is all there is. The "cold logician" Parmenides has been compared to *ice*, and Heraclitus' thinking is the *fiery* one [29]. If Hersch is right, and this opposition between these styles crosses the history of philosophy like a red line, then this must be true no less for the history of *science*.

A textbook example illustrating the described antagonism is the debate between *Newton* and *Leibniz* [39]: For Newton, space and time are fundamental and given *a priori*, just like a stage on which all the play is located. For Leibniz, on the other hand, space and time are emergent as *relational* properties: The stage emerges *with* the play and not prior to it, and it is not there without it. With only a few exceptions — most notably *Ernst Mach* — the course of physical science went for Newton's view; it did so with good success. An important example here is, of course, Einstein's relativity: Whilst its crystallization point was *Mach's principle*, stating that intertial forces are relational (as opposed to coming from acceleration against an *absolute* space), the resulting theory does *not* follow the principle since there is (the flat) space-time also in a massless universe.

In the present work, we turn our attention to physical phenomena such as the second law of thermodynamics and Bell correlations from quantum theory. We find here again the opposition between Parmenides' and Heraclitus' standpoints, and we directly build on their tension with the goal of obtaining more insight, hereby bridging the gap separating them to some extent.

The *Heraclitean* style can be recognized again in the spirit of *Ferdinand Gonseth*'s "La logique est tout d'abord une science naturelle" — "Logic is, first of all, a natural science." This is a predecessor of Rolf Landauer's famous slogan "Information Is Physical," [27] putting physics at the basis of the concept of information and its treatment.

This is in sharp contrast to *Shannon*'s [34] (very successful) making information *abstract*, independent of the particular physical realization of it (*e.g.*, a specific noisy communication channel). To the *Parmenidean* paradigm belongs also the *Church-Turing hypothesis* [25], stating that all physically possible processes can be simulated by a universal Turing machine. This basing physical reality on information and computation was later summarized by *John Archibald Wheeler* as "It from Bit" [40].

1.2 Non-Locality, Space-Time Causality, and Randomness

After Einstein had made the world mechanistic and "local" (without actions at a distance), he was himself involved in a work [17] paving the (long) way to that locality to fall again. The goal of Einstein *et al.* had, however, been the exact opposite: to *save* locality in view of correlations displayed in the measurement behavior of (potentially physically separated) parts of an *entangled* quantum state. The claim was that quantum theory was an only incomplete description of nature, to be refined by hidden parameters determining the outcomes of all potential, alternative measurements. It took roughly thirty years until that claim was grounded when *John Stewart Bell* [7] showed the impossibility of the program — ironically making the case with the exact same states as "EPR" had introduced. The consequences of Bell's insight are radical: If the values are not predetermined, then there must be *fresh* and at the same time *identical* pieces of classical information popping us spontaneously — this is *non-locality.* The conceptual problem these correlations lead us into is the difficulty of explaining their origin *causally*, *i.e.*, according to *Reichenbach's principle* — which states that a correlation between two space-time events can stem from a *common cause* (in the common past) or a *direct influence* from one event to the other [33]. Bell's result rules out the common cause as an explanation, thus remains the influence. Besides the fact that it is an inelegant overkill to explain a *non*-signaling phenomenon (*not* allowing for transmitting messages from one party to the other) using a *signaling* mechanism, there are further problems: Hidden influences as explanations of Bell correlation require both infinite speed [36], [16], [1] and precision [42].

In view of this, it appears reasonable to question the (only) assumption made in Reichenbach's principle: The *a priori* causal structure [30].[1] If we turn back the wheel to the *Newton-Leibniz debate*, and choose to follow Leibniz instead, seeing space-time as appearing only *a posteriori*, then there is a first victim to this: *Randomness*: In [15], a piece of information is called *freely random* if it is statistically independent from all other pieces of information except the ones in its *future* light cone. Clearly, when the assumption of an initially given causal structure is dropped, such a definition is not founded any longer.[2] (It is then possible to turn around the affair and base past and future on postulated freeness of bits [6].) In any case, we are now motivated to find an *intrinsic, context-free, physical definition of randomness* and choose to look at: *Watt's Steam Engine.*

2 The Search for an Intrinsic Randomness Notion: From Steam Pipes to the Second Law of Thermodynamics

2.1 The Fragility and the Robustness of the Second Law

The *second law of thermodynamics* has advanced to becoming pop culture.[3] It is, however, much less famous than Einstein's relativity, Heisenberg's uncertainty, or quantum teleportation because it does not have any glamour, fascination, or hope attached to it: The law stands for facts many of us are in denial of or try to escape. We ask whether the attribution of that formalized pessimism to physics has *historical* reasons.

The validity of the second law seems to depend on surprising conditions such as the inexistence of certain life styles (*e.g.*, *Maxwell's demon* or photo-synthesizing plants — Kelvin [24] writes: "When light is absorbed *other than in vegetation*, there is dissipation [...]"). To make things worse, there is always a non-zero probability (exponentially small, though) of exceptions where the law fails to hold; we are not used to this from other laws of physics. Can this be taken as an indication that the fundamental way of formulating the law eludes us?

The described *fragility* of the second law is strangely contrasted by its being, in another way, *more robust* than others (such as Bell violations only realizable under extremely precise lab conditions): We certainly do not need to trust experimentalists to be convinced that the second law is acting, everywhere and always. It has even been claimed [38] to hold a "supreme position" among physical laws: It appears easier to imagine a world where relativity or quantum theory

[1] It has been shown [5] that if causality is dropped but logical consistency maintained, then a rich world opens — comparable to the one between locality and signaling.

[2] Note furthermore that the definition is consistent with full determinism: A random variable with trivial distribution is independent of every other (even itself).

[3] See, *e.g.*, Allen, W., Husbands and Wives (1992): The protagonist Sally is explaining why her marriage did not work out. First she does not know, then she realizes: "It's the *second law of thermodynamics*: sooner or later everything turns to shit. That's my phrasing, not the *Encyclopedia Britannica*."

do not hold than to figure out a reality lacking the validity of the second law. (Concerning the reasons for this, we can only speculate: Would we be forced to give up the mere *possibility of perception, memory — the arrow of time*?)

2.2 History

This story (see [38]) starts with *Sadi Carnot* (1796–1832) and his study of heat engines such as *James Watt*'s steam pipe. The assumption, in consequence, that the law is closely related to such engines, and to the circular processes involved, is of course not wrong, but it underestimates a fundamental *logical-combinatorial-informational fact*; perhaps steam engines are to the second law what *telescopes* are for Jupiter's moons.

Carnot argued that the maximal efficiency of a heat engine between two heat baths depended only on the two temperatures involved. (The derived formula motivated Lord Kelvin to define the absolute temperature scale.)

Rudolf Clausius' (1822–1888) [14] version of the second law reads: *"Es kann nie Wärme aus einem kälteren in einen wärmeren Körper übergehen, ohne dass eine andere damit zusammenhängende Änderung eintritt."* — "No process can transport heat from cold to hot and do no further change."

Lord Kelvin (1824–1907) [24] formulated his own version of the second law and concluded — in just the next sentence — that the law may have consequences deeper than what was obvious at first sight: *"Restoration of mechanical energy without dissipation [...] is impossible. Within a finite period of time past, the earth must have been, within a finite time, the earth must again be unfit for the habitation of man."*

Also for Clausius, it was only a single thinking step from his version of the law to concluding that all temperature differences in the entire universe will vanish (the *Wärmetod*) and that then, no change will be possible anymore. He speaks of *a general tendency of nature for change into a specific direction:* "Wendet man dieses auf das Weltall im Ganzen an, so gelangt man zu einer eigentümlichen Schlussfolgerung, auf welche zuerst W. Thomson [Lord Kelvin] aufmerksam machte, *nachdem er sich meiner Auffassung des zweiten Hauptsatzes angeschlossen hatte*. Wenn [...] im Weltall die Wärme stets das Bestreben zeigt, [...] dass [...] Temperaturdifferenzen ausgeglichen werden, so muss es sich mehr und mehr dem Zustand annähern, wo [...] keine Temperaturdifferenzen mehr existieren." — in short: "He was right after he had realized that I had been right: At the end, no temperature differences will be left in the universe."

Ludwig Boltzmann (1844–1906) brought our understanding of the second law closer to combinatorics and probability theory (in particular, the law of large numbers). His version is based on the fact that it is more likely to end up in a large set (of possible states) than in a small one: The more "microstates" belong to a given "macrostate," the more likely is it that you will find yourself in that macrostate. In other words, if you observe the time evolution of a system (by some reason starting in a very small, "unlikely" macrostate), then the "entropy" of the system — here simply (the logarithm of) the number of corresponding microstates — does not decrease.[4]

The notion of macrostate and its entropy have been much debated. Von Neumann remarked [38]: “No one knows what entropy really is, so in a debate you will always have the advantage.” We aim at a version of the second law avoiding this advantage: a view without probabilities or ensembles, but based on intrinsic, one-shot complexity instead. Crucial steps in that direction were made by *Zurek* [43]. We take a Church-Turing view and follow *Landauer* [27] whose role or, more specifically, whose choice of viewpoint around the second law can be compared with *Ernst Specker*'s [35] take on quantum theory: *logical.*

2.3 Reversibility

Landauer investigated the thermodynamic price of logical operations. He was correcting a belief by *John von Neumann* that every bit operation required free energy $kT\ln 2$ (where k is Boltzmann's constant, T the environmental temperature, and $\ln 2$ owed to the fact that 2 is not a natural but a logical constant). According to Landauer — and affirmed by *Fredkin and Toffoli's* “ballistic computer” [21] —, this limitation or condition only concerns (bit) operations which are logically *irreversible*, such as the AND or the OR. On the positive side, it has been observed that every function, bijective or not, can in principle be evaluated in a logically *reversible way, using only “Toffoli gates,” i.e., made-reversible and then-universal AND gates*; its computation can be thermodynamically neutral: It does not have to dissipate heat.

Landauer's principle states erasing (setting the corresponding memory cells to 0) N bits costs $kTN\ln 2$ free energy which must be dissipated as heat to the environment (of temperature T). This *dissipation* is crucial in the argument: Heating up the environment compensates for the *entropy loss* within the memory cell, realized as a physical system (spin, gas molecule, *etc.*).

Let us consider the inverse process: *Work extraction.* Bennett [8] made the key contribution to the resolution of the paradox of *Maxwell's demon.* That demon had been thought of as violating the second law by adaptively handling a frictionless door with the goal of “sorting a gas” in a container. Bennett took the demon's memory (imagined to be in the all-0-state before sorting) into account, which is in the end filled with “random” information, an expression of the original state of the gas. The growth of disorder *inside* the demon compensates for the

[4] Boltzmann imagined further that the universe had started in a completely “uniform” state, so the entire, rich reality perceived would be a simple fluctuation. (Note that the fact that this fluctuation is extremely unlikely is irrelevant if we can *condition on our existence*, given our discussing this.) He may have been aware that this way of thinking leads straight into *solipsism:* “My existence alone, simply *imagining* my environment, seems much more likely than the actual existence of all people around me, let alone all the visible galaxies, *etc.*” — he killed himself in a hotel room in Duino, Italy; it has been told that this was also related to “mobbing” by Mach in Vienna. In any case, we choose to comfort us today with the somewhat religious assumption that the universe initiated in a low-entropy state, called the *big bang.*

order she creates *outside* (*i.e.*, in the gas) — the second law is saved. The initial 0-string is the demon's resource allowing for her order creation.

If we break Bennett's argument apart in the middle, we end up with the *converse* of Landauer's principle: The all-0-string has work value, *i.e.*, if we accept the price of the respective memory cells to become "randomized" in the process, we can extract $kTN \ln 2$ free energy from the environment (a heat bath of temperature T). In a *constructivist* manner, we choose to view the work-extraction process as an algorithm which, according to the *Church-Turing hypothesis,* we imagine as being carried out by a universal *Turing machine*. It then follows that the *work value of a string* S is closely related to the possibility of lossless compression of that string: For any concrete data-compression algorithm, we can extract $kT \ln 2$ times the length of S (uncompressed) minus the length of its compression: *Work value is redundancy (in representation) of information.* On the other end of the scale, the upper bound on work extraction is linked to the ultimate compression limit: *Kolmogorov complexity*, *i.e.*, the length of the shortest program for the extraction demon (Turing machine) generating the string in question. This holds because a computation is logically reversible only if it can be carried out in the other direction, step by step.

There is a direct connection between the work value and the *erasure cost* (in the sense of Landauer's principle) of a string. We assume here that for both processes, the extraction demon has access to an additional string X (modeling prior "knowledge" about S) which serves as a catalyst and is to be unchanged at at the end of the process. For a string $S \in \{0,1\}^N$, let $\mathrm{WV}(S|X)$ and $\mathrm{EC}(S|X)$ be its work value and erasure costs, respectively, given X. Then[5]

$$\mathrm{WV}(S|X) + \mathrm{EC}(S|X) = N \ .$$

To see this, consider first the combination extract-then-erase. Since this is *one specific way* of erasing, we have

$$\mathrm{EC}(S|X) \leq N - \mathrm{WV}(S|X) \ .$$

If, on the other hand, we consider the combination erase-then-extract, this leads to

$$\mathrm{WV}(S|X) \geq N - \mathrm{EC}(S|X) \ .$$

Given the results on the work value discussed above, as well as this connection between the work value and erasure cost, we obtain the following bounds on the thermodynamic cost of erasing a string S by a demon, modeled as a universal Turing machine $\mathcal{U}$ with initial tape content X.

Landauer's principle, revisited. *Let C be a computable compression function*

$$C \,:\, \{0,1\}^* \times \{0,1\}^* \longrightarrow \{0,1\}^*$$

such that $(A,B) \mapsto (C(A,B),B)$ is injective. Then we have

$$K_{\mathcal{U}}(S|X) \leq \mathrm{EC}(S|X) \leq \mathrm{len}(C(S,X)) \ .$$

[5] Let $kT \ln 2 = 1$.

Landauer's revised principle puts forward two ideas: First, the erasure cost is an *intrinsic, context-free, physical measure for randomness* (entirely independent of probabilities and counter-factual statements of the form "some value *could* just as well have been *different*," *i.e.*, removing one layer of speculation). The idea that the erasure cost — or the Kolmogorov complexity related to it — is a measure for randomness independent of probabilities can be tested in a context in which randomness has been paramount: *Bell correlations* [7] predicted by quantum theory, see Section for details 3.

The second idea starts from the observation that the price for the *logical* irreversibility of the erasure transformation comes in the form of a *thermodynamic* effort.[6] In an attempt to harmonize this somewhat *hybrid* picture, we invoke Wheeler's [40] *"It from Bit*: Every *it* — every particle, every field of force, even the space-time continuum itself — derives its function, its meaning, its very existence entirely [...] from the apparatus-elicited answers to yes-or-no questions, binary choices, *bits*." This is an anti-thesis to Landauer's slogan, and we propose the following synthesis of the two: If Wheeler suggests to look at the environment as being *information* as well, then Landauer's principle ends up to be read as: The necessary environmental compensation for the logical irreversibility of the erasure of S is such that *the overall computation, including the environment, is logically reversible: no information ever gets completely lost.*

Second law, Church-Turing view. *If reality is assumed to be computed by a Turing machine, then that computation has the property of being injective: Nature computes with Toffoli, but no AND or OR gates.*

This fact is *a priori a*symmetric in time: The future must uniquely determine the past, not necessarily *vice versa.* (This is identical with *Grete Herrmann's* [22] take on causality.) In case the condition holds also for the reverse time direction, the computation is *deterministic*, and *randomized* otherwise.

2.4 Consequences

If logical reversibility is a simple computational version of a discretized second law, does it have implications resembling the traditional versions of the law?

Logical reversibility implies quasi-monotonicity.

First of all, we find a "Boltzmann-like" form, *i.e.*, the existence of a quantity essentially monotonic in time. More specifically, the logical reversibility of time evolution implies that the Kolmogorov complexity of the global state at time t

[6] Since the amount of the required free energy (and heat dissipation) is proportional to the length of the best compression of the string, the latter can be seen as a *quantification* of the erasure transformation's irreversibility.

can be smaller than the one at time 0 only by at most $K(C_t) + O(1)$ if C_t is a string encoding the time span t. The reason is that one possibility of describing the state at time 0 is to give the state at time t, plus t itself; the rest is exhaustive search using only a constant-length program simulating forward time evolution (including possible randomness).

Logical reversibility implies Clausius-like law.

Similarly, logical reversibility also implies statements resembling the version of the second law due to *Clausius*: "Heat does not spontaneously flow from cold to hot." The rationale here is that if we have a computation — the time evolution — using only (logically reversible) Toffoli gates, then it is *impossible* that this circuit computes a transformation mapping a pair of strings to another pair such that the Hamming-heavier of the two becomes even heavier whilst the lighter gets lighter. A function *accentuating* imbalance, instead of lessening it, is not reversible, as a basic counting argument shows.

Example. Let a circuit consisting of only Toffoli gates map an $N(= 2n)$-bit string to another. We consider the map separately on the first and second halves and assume the computed function to be conservative, *i.e.*, to leave the Hamming weight of the full string unchanged at n (conservativity can be seen as some kind of *first* law, *i.e.*, the preservation of a quantity). We look at the excess of 1's in one of the halves (which equals the deficit of 1's in the other). We observe that the probability (with respect to the uniform distribution over all strings of some Hamming-weight couple $[wn, (1-w)n]$) of the *imbalance substantially growing* is exponentially weak. The key ingredient for the argument is the function's injectivity. Explicitly, the probability that the weight couple changes from $[wn, (1-w)n]$ to $[(w + \Delta)n, (1 - w - \Delta)n]$ — or more extremely —, for $1/2 \leq w < 1$ and $0 < \Delta \leq 1 - w$, is

$$\frac{\binom{n}{(w+\Delta)n}\binom{n}{(1-w-\Delta)n}}{\binom{n}{wn}\binom{n}{(1-w)n}} = 2^{-\Theta(n)} \ .$$

Note here that we even get the correct, exponentially weak "error probability" with which the traditional second law can be "violated."

Logical reversibility implies Kelvin-like law.

"A single heat bath alone has no work value." This, again, follows from a simple counting argument. There exists no reversible circuit that, for general input environments (with a fixed weight — intuitively: *heat energy*), extracts redundancy, *i.e.*, work value, and concentrates it in some pre-chosen bit positions: *Concentrated* redundancy is *more* of it.

Example. The probability that a fixed circuit maps a "Hamming bath" of length N and Hamming weight w to another such that the first n positions contain all 1's and such that the Hamming weight of the remaining $N - n$ positions is $w - n$ (again, we are assuming conservation here) is

$$\frac{\binom{N-n}{w-n}}{\binom{N}{w}} = 2^{-\Theta(n)} \ .$$

2.5 Discussion and Questions

We propose a logical view of the second law of thermodynamics: *the injectivity or logical reversibility of time evolution.* This is somewhat ironic as the second law has often been related to its exact opposite: *irreversibility.*[7] It implies, within the Church-Turing view, Clausius-, Kelvin-, and Boltzmann-like statements. We arrive at seeing a law *combinatorial in nature* — and its discovery in the context of steam pipes as a historical incident.

A logically reversible computation can still split up paths [18].[8] This "randomness" may bring in *objective* time asymmetry. What is then the exact mechanism by which randomness implies that a *record* tells more about the past than about the future? (Does it?)

3 Bell Correlations and the Church-Turing Hypothesis

We test the obtained intrinsic notion of randomness, in the form of erasure cost or Kolmogorov complexity, with a physical phenomenon that we have already mentioned above as challenging *a-priori* causality: *"non-local" correlations* from quantum theory. In fact, randomness has been considered crucial in the argument. We put this belief into question in its exclusiveness; at the same time we avoid in our reasoning connecting results of different measurements that, in fact, exclude each other (in other words, we refrain from assuming so-called *counterfactual definiteness*, *i.e.*, that all these measurement outcomes even *exist* alto-

[7] Since new randomness cannot be gotten rid of later, the equation reads: "Logical reversibility plus randomness equals thermodynamic *ir*reversibility." If you *can* go back logically in a random universe, then you certainly *cannot* thermodynamically.

[8] Note that there is no (objective) splitting up, or randomness, if time evolutions are unitary, *e.g.*, come from Schrödinger, heat-propagation, or Maxwell's equations. What is then the origin of the arrow of time? The quantum-physical version of injectivity is *Hugh Everett III's relative-state interpretation.* How do we imagine the bridge from global unitarity to the subjective perception of time asymmetry? When we looked above, with Landauer, at a closed *classical* system of two parts, then the (possible) complexity deficit in one of them must simply be compensated in a corresponding increase in the other. In Everett's view, this means that there can be low-entropy *branches of the wave function* (intuitively, yet too naïvely, called: parallel universes) as long as they are compensated by other, highly complex ones.

gether).[9] For the sake of comparison, we first review the common, probabilistic, counter-factual reasoning.

3.1 Bell Non-Local Correlations

Non-locality, manifested in violations of *Bell inequalities*, expresses the impossibility to prepare parts of an entangled system simultaneously for *all possible measurements*. We look at an idealized non-local correlation, the *Popescu-Rohrlich (PR) box* [31]. Let A and B be the respective input bits to the box and X and Y the output bits; the (classical) bits satisfy

$$X \oplus Y = A \cdot B\,. \qquad (1)$$

According to a result by Fine [20], the non-locality of the system (*i.e.*, conditional distribution) $P_{XY|AB}$, which means that it cannot be written as a convex combination of products $P_{X|A} \cdot P_{Y|B}$, is equivalent to the fact that there exists no preparation for all alternative measurement outcomes $P'_{X_0X_1Y_0Y_1}$ such that

$$P'_{X_iY_j} = P_{XY|A=i,B=j}$$

for all $(i, j) \in \{0, 1\}^2$. In this view, non-locality means that the outputs cannot *exist*[10] before the inputs do. Let us make this qualitative statement more precise. We assume a perfect PR box, *i.e.*, a system always satisfying $X \oplus Y = A \cdot B$. Note that this equation alone does not uniquely determine $P_{XY|AB}$ since the marginal of X, for instance, is not determined. If, however, we additionally require *no-signaling*, then the marginals, such as $P_{X|A=0}$ or $P_{Y|B=0}$, must be perfectly unbiased under the assumption that all four (X, Y)-combinations, *i.e.*, $(0, 0), (0, 1), (1, 0)$, and $(1, 1)$, are possible. To see this, assume on the contrary that $P_{X|A=0,B=0}(0) > 1/2$. By the PR condition (1), we can conclude the same for Y: $P_{Y|A=0,B=0}(0) > 1/2$. By no-signaling, we also have $P_{X|A=0,B=1}(0) > 1/2$. Using symmetry, and no-signaling again, we obtain both $P_{X|A=1,B=1}(0) > 1/2$ and $P_{Y|A=1,B=1}(0) > 1/2$. This contradicts the PR condition (1) since *two*

[9] The *counter-factual* nature of the reasoning claiming "non-classicality" of quantum theory, that was the main motivation in [41], has already been pointed out by Specker [35]: "In einem gewissen Sinne gehören aber auch die scholastischen Spekulationen über die *Infuturabilien* hieher, das heisst die Frage, ob sich die göttliche Allwissenheit auch auf Ereignisse erstrecke, die eingetreten wären, falls etwas geschehen wäre, was nicht geschehen ist." — "In some sense, this is also related to the scholastic speculations on the *infuturabili*, *i.e.*, the question whether divine omniscience even extends to what would have happened if something had happened that did not happen."

[10] What does it mean that a classical bit *exists*? Note first that *classicality* of information implies that it can be measured without disturbance and that the outcome of a "measurement" is always the same; this makes it clear that it is an *idealized* notion requiring the classical bit to be represented in a redundant way over an *infinite* number of degrees of freedom, as a thermodynamic limit. It makes thus sense to say that a *classical bit U exists*, *i.e.*, has taken a definite value.

bits which are both biased towards 0 *cannot differ with certainty.* Therefore, our original assumption was wrong: The outputs *must* be perfectly unbiased. Altogether, this means that X as well as Y cannot exist (*i.e.*, take a definite value — actually, there cannot even exist a classical value arbitrarily weakly correlated with one of them) *before* the classical bit $f(A, B)$ exists for some nontrivial deterministic function $f : \{0, 1\}^2 \to \{0, 1\}$. The paradoxical aspect of non-locality — at least if a causal structure is in place — now consists of the fact that *fresh* pieces of information *come to existence* in a *spacelike-separated* manner that are nonetheless *perfectly correlated*.

3.2 Kolmogorov Complexity

We introduce the basic notions required for our alternative, complexity-based view. Let $\mathcal{U}$ be a fixed universal Turing machine (TM).[11] For a finite or infinite string s, the *Kolmogorov complexity* [26], [28] $K(s) = K_{\mathcal{U}}(s)$ is the length of the shortest program for $\mathcal{U}$ such that the machine outputs s. Note that $K(s)$ can be infinite if s is.

Let $a = (a_1, a_2, \ldots)$ be an infinite string. Then

$$a_{[n]} := (a_1, \ldots, a_n, 0, \ldots) \ .$$

We study the asymptotic behavior of $K(a_{[n]}) : \mathbf{N} \to \mathbf{N}$. For this function, we simply write $K(a)$, similarly $K(a \,|\, b)$ for $K(a_{[n]} \,|\, b_{[n]})$, the latter being the length of the shortest program outputting $a_{[n]}$ upon input $b_{[n]}$. We write

$$K(a) \approx n \ :\Longleftrightarrow \ \lim_{n\to\infty} \frac{K(a_{[n]})}{n} = 1 \ .$$

We call a string a with this property *incompressible*. We also use $K(a_{[n]}) = \Theta(n)$, as well as

$$K(a) \approx 0 :\Longleftrightarrow \lim_{n\to\infty} \frac{K(a_{[n]})}{n} = 0 \Longleftrightarrow K(a_{[n]}) = o(n) \ .$$

Note that *computable* strings a satisfy $K(a) \approx 0$, and that incompressibility is, in this sense, the extreme case of uncomputability.

Generally, for functions $f(n)$ and $g(n) \not\approx 0$, we write $f \approx g$ if $f/g \to 1$. *Independence of a and b* is then[12]

$$K(a \,|\, b) \approx K(a)$$

or, equivalently,

$$K(a, b) \approx K(a) + K(b) \ .$$

[11] The introduced asymptotic notions are independent of this choice.

[12] This is inspired by [12] (see also [13]), where (joint) Kolmogorov complexity — or, in practice, any efficient compression method — is used to define a *distance measure* on sets of bit strings (such as literary texts of genetic information of living beings). The resulting structure in that case is a distance measure, and ultimately a clustering as a binary tree.

If we introduce

$$I_K(x;y) := K(x) - K(x\,|\,y) \approx K(y) - K(y\,|\,x)\ ,$$

independence of a and b is $I_K(a,b) \approx 0$.

In the same spirit, we can define *conditional independence*: We say that *a and b are independent given c* if

$$K(a,b\,|\,c) \approx K(a\,|\,c) + K(b\,|\,c)$$

or, equivalently,

$$K(a\,|\,b,c) \approx K(a\,|\,c)\ ,$$

or

$$I_K(a;b\,|\,c) := K(a\,|\,c) - K(a\,|\,b,c) \approx 0\ .$$

3.3 Correlations and Computability

We are now ready to discuss non-local correlations with our context-free randomness measure. The mechanism we discover is very similar to what holds probabilistically: If the choices of the measurements are random (uncomputable) and non-signaling holds, then the outputs must be random (uncomputable) as well. We prove the following statement.

Uncomputability from Correlations. *There exist bipartite quantum states with a behavior under measurements such that if the sequences of setting encodings are maximally uncomputable (incompressible), then the sequences of measurement results are uncomputable as well, even given the respective setting sequences.*

Proof. We proceed step by step, starting with the idealized system of the PR box. Let first (a,b,x,y) be infinite binary strings with

$$x_i \oplus y_i = a_i \cdot b_i\ . \tag{2}$$

Obviously, the intuition is that the strings stand for the inputs and outputs of a PR box. Yet, no dynamic meaning is attached to the strings anymore (or to the “box,” for that matter) since there is no *“free choice”* of an input and no generation of an output in function of the input; all we have is a quadruple of strings satisfying the PR condition (2). However, nothing prevents us from defining this (static) situation to be *no-signaling*:

$$K(x\,|\,a) \approx K(x\,|\,ab) \quad \text{and} \quad K(y\,|\,b) \approx K(y\,|\,ab)\ . \tag{3}$$

We argue that if the inputs are incompressible and independent, and no-signaling holds, then the outputs must be uncomputable: To see this, assume now that $(a,b,x,y) \in (\{0,1\}^{\mathbf{N}})^4$ with $x \oplus y = a \cdot b$ (bit-wisely), no-signaling (3), and

$$K(a,b) \approx 2n\ ,$$

i.e., the "input" pair is incompressible. We conclude

$$K(a \cdot b \,|\, b) \approx n/2 \ .$$

Note first that $b_i = 0$ implies $a_i \cdot b_i = 0$, and second that any further compression of $a \cdot b$, given b, would lead to "structure in (a, b)," *i.e.*, a possibility of describing (programming) a given b in shorter than n and, hence, (a, b) in shorter than $2n$. Observe now

$$K(x \,|\, b) + K(y \,|\, b) \geq K(a \cdot b \,|\, b)$$

which implies

$$K(y \,|\, b) \geq K(a \cdot b \,|\, b) - K(x \,|\, b) \geq n/2 - K(x) \ . \tag{4}$$

On the other hand,

$$K(y \,|\, a, b) \approx K(x \,|\, a, b) \leq K(x) \ . \tag{5}$$

Now, no-signaling (3) together with (4) and (5) implies

$$n/2 - K(x) \leq K(x) \ ,$$

and

$$K(x) \geq n/4 = \Theta(n) \ :$$

(This bound can be improved by a more involved argument [4].) The string x must be uncomputable.

A priori, it is not overly surprising to receive uncomputable outputs upon inputs having the same property. Thus, we now turn our attention to the *conditional* output complexities *given* the inputs: We consider the quantities $K(x \,|\, a)$ and $K(y \,|\, b)$. Note first

$$K(x \,|\, a) \approx 0 \Leftrightarrow K(x \,|\, ab) \approx K(y \,|\, ab) \approx 0 \Leftrightarrow K(y \,|\, b) \approx 0 \ ,$$

i.e., the two expressions vanish simultaneously. We show that, in fact, they both fail to be of order $o(n)$. To see this, assume $K(x \,|\, a) \approx 0$ and $K(y \,|\, b) \approx 0$. Hence, there exist programs P_n and Q_n (both of length $o(n)$) for functions f_n and g_n with

$$f_n(a_n) \oplus g_n(b_n) = a_n \cdot b_n \ . \tag{6}$$

For fixed (families of) functions f_n and g_n, asymptotically how many (a_n, b_n) can at most exist that satisfy (6)? The question boils down to a *parallel-repetition* analysis of the *PR game*: A result by Raz [32] implies that the number is of order $(2 - \Theta(1))^{2n}$. Therefore, the two programs P_n and Q_n together with the index, of length

$$(1 - \Theta(1))2n \ ,$$

of the correct pair (a, b) within the list of length $(2 - \Theta(1))^{2n}$ lead to a program, generating (a, b), that has length

$$o(n) + (1 - \Theta(1))2n \ ,$$

in contradiction to the assumption of incompressibility of (a, b).

Unfortunately, perfect PR boxes are not predicted by quantum theory. We show that correlations which *are* achievable in the laboratory [37] allow for the argument to go through; they are based on the *chained Bell inequality* [3] instead of perfect PR-type non-locality.

To the chained Bell inequality belongs the following idealized system: Let $A, B \in \{1, \ldots, m\}$ be the inputs. We assume the "promise" that B is congruent to A or to $A+1$ modulo m. Given this promise, the outputs $X, Y \in \{0, 1\}$ must satisfy

$$X \oplus Y = \chi_{A=m,B=1} \ , \tag{7}$$

where $\chi_{A=m,B=1}$ is the characteristic function of the event $\{A = m, B = 1\}$.

Barrett, Hardy, and Kent [3] showed that if A and B are random, then X and Y must be perfectly unbiased if the system is no-signaling. More precisely, they were even able to show such a statement from the gap between the error probabilities of the best classical — $\Theta(1/m)$ — and quantum — $\Theta(1/m^2)$ — strategies for winning this game.

We assume $(a, b, x, y) \in (\{1, \ldots, m\}^n)^2 \times (\{0, 1\}^n)^2$ to be such that the promise holds, and such that

$$K(a, b) \approx (\log m + 1) \cdot n \ , \tag{8}$$

i.e., the string $a||b$ is maximally incompressible given the promise; the system is no-signaling (3); the fraction of quadruples (a_i, b_i, x_i, y_i), $i = 1, \ldots, n$, satisfying (7) is of order $(1 - \Theta(1/m^2))n$. Then $K(x) = \Theta(n)$.

To see this, observe first that $K(a, b)$ being maximal implies

$$K(\chi_{a=m,b=1} \,|\, b) \approx \frac{n}{m} \ : \tag{9}$$

The fractions of 1's in b must, asymptotically, be $1/m$ due to the string's incompressibility. If we condition on these positions, the string $\chi_{a=m,b=1}$ is incompressible, since otherwise there would be the possibility of compressing (a, b).

Now, we have

$$K(x \,|\, b) + K(y \,|\, b) + h(\Theta(1/m^2))n \geq K(\chi_{a=m,b=1} \,|\, b)$$

since one possibility for "generating" the string $\chi_{a=m,b=1}$, from position 1 to n, is to generate $x_{[n]}$ and $y_{[n]}$ as well as the string indicating the positions where (7) is violated, the complexity of the latter being at most[13]

$$\log \binom{n}{\Theta(1/m^2)n} \approx h(\Theta(1/m^2))n \ .$$

Let us compare this with $1/m$: Although the binary entropy function has slope ∞ in 0, we have

$$h(\Theta(1/m^2)) < 1/(3m)$$

[13] Here, h is the binary entropy $h(x) = -p \log_2 p - (1-p) \log_2(1-p)$. Usually, p is a probability, but h is invoked here merely as an approximation for binomial coefficients.

if m is sufficiently large. To see this, observe first that the dominant term of $h(x)$ for small x is $-x \log x$, and second that

$$c(1/m) \log(m^2/c) < 1/3$$

for m sufficiently large.

Together with (9), we now get

$$K(y \mid b) \geq \frac{2n}{3m} - K(x) \tag{10}$$

if m is chosen sufficiently large. On the other hand,

$$K(y \mid ab) \leq K(x \mid ab) + h(\Theta(1/m^2))n \tag{11}$$
$$\leq K(x) + \frac{n}{3m} \ . \tag{12}$$

Now, (3), (10), and (12) together imply

$$K(x) \leq \frac{n}{6m} = \Theta(n) \ ;$$

in particular, x must be uncomputable. This concludes the proof. □

3.4 Kolmogorov Amplification and the All-or-Nothing Nature of the Church-Turing Hypothesis

The shown result implies that *if* the experimenters are given access to an incompressible number (such as Ω [11]) for choosing their measurement bases, *then* the measured photon (in a least one of the two labs) is forced to generate an uncomputable number as well, even given the string determining its basis choices.

This is a similar observation as in the probabilistic realm, where certain "free-will theorems" have been formulated in the context. In fact, stronger statements hold there, since non-local correlations allow for *randomness amplification as well as expansion* (see, *e.g.*, [15]): The randomness generated by the photons as their measurement output qualitatively *and* quantitatively *exceeds* what is required for the choices of the measurement settings. This also holds in our complexity-based model: Indeed, it has been shown in [4] that functionalities such as *Kolmogorov-complexity amplification and expansion* are possible using Bell correlations. The consequence is that there is either no incompressibility or uncomputability at all in the world, or it is full of it.

All-or-Nothing Feature of the Church-Turing Hypothesis. *Either no device exists in nature allowing for producing uncomputable sequences, or even a single photon can do it.*

4 Concluding Remarks and Open Questions

The antagonism between the pre-Socratic philosophers *Parmenides* and *Heraclitus* is still vivid in today's thinking traditions: The Parmenidean line puts *logic* is the basis of space-time and dynamics — in the end all of physics. It has inspired researchers such as Leibniz, Mach, or Wheeler. Central here is a doubt about *a priori* absolute space-time causality: Is it possible that these concepts only emerge at a higher level of complexity, along with macroscopic, classical information?

Fundamentally opposed is the Heraclitean style, seeing *physics and its objects* at the center: space, time, causality, and dynamic change is what all rests upon, including logic, computation, or information. To this tradition belong Newton, most physicists including Einstein, the logician Gonseth, certainly Landauer.

According to *Paul Feyerabend* [19], a specific tradition comes with its own criteria for success *etc.*, and it can be judged from the standpoint of another (with those other criteria). In this spirit, it has been the goal of our discourse to build bridges between styles, and to use their tension to serve us. This allowed, for instance, to get more insight into the second law of thermodynamics or the "non-local" correlations from quantum theory. The latter challenge our established views of space and time; they, actually, have us look back to the debate between Newton and Leibniz and to question the path most of science decided to take, at that time.

For the sake of a final thought, assume *à la* Leibniz that space, time, and causality do not exist prior to *classical information* — which we understand as an idealized notion of *macroscopically and highly redundantly represented* information; an ideal classical bit can then be measured without disturbance, copied, and easily recognized as being classical. In this view, classicality is a *thermodynamic* notion. Thus the key to the *quantum measurement process*, and the problems linked to it, may lie within thermodynamics. (Yet, even if this is successful: How come we observe correlations of pieces of *classical* information unexplainable by any reasonable *classical* mechanism? How can quantum correlations and thermodynamic classicality — *Bell & Boltzmann* — be reconciled?)

Acknowledgments. This text is based on a presentation at the *14th Annual Conference on Theory and Applications of Models of Computation (TAMC 2017)* at the Universität Bern. I am grateful to Gerhard Jäger and all the organizers for kindly inviting me to give a talk.

I thank Mateus Araújo, Veronika Baumann, Ämin Baumeler, Charles Bédard, Claus Beisbart, Gilles Brassard, Harvey Brown, Caslav Brukner, Harry Buhrman, Matthias Christandl, Sandro Coretti, Fabio Costa, Bora Dakic, Frédéric Dupuis, Paul Erker, Adrien Feix, Jürg Fröhlich, Manuel Gil, Nicolas Gisin, Esther Hänggi, Arne Hansen, Marcus Huber, Lorenzo Maccone, Alberto Montina, Samuel Ranellucci, Paul Raymond-Robichaud, Louis Salvail, L. Benno Salwey, Martin Schüle, Andreas Winter, and Magdalena Zych for inspiring discussions, and the *Pläfä-Einstein* as well as the *Reitschule* for their inspiring atmosphere.

This research is supported by the Swiss National Science Foundation (SNF), the National Centre of Competence in Research “Quantum Science and Technology” (QSIT), the COST action on Fundamental Problems in Quantum Physics, and by the *Hasler Foundation.*

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