Pseudo-Telepathy, Entanglement, and Graph Colorings

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Abstract — **Quantum entanglement allows for a phenomenon called pseudo-telepathy. This can be used to devise a simple experiment demonstrating the existence of quantum entanglement, provided that the parameters are chosen such that it is impossible to win the game without entanglement or communication (or telepathy). We describe a close connection between pseudo-telepathy and graph coloring and use this to correct previous beliefs on the possibility of doing pseudo-telepathy.**

I. Pseudo-Telepathy...

Consider the following game, called *pseudo-telepathy*: Two players who cannot communicate are asked two separate questions. In order to win the game, their answers to these questions have to be equal if and only if the questions were equal. More specifically, the questions asked to the players are two N-bit strings v_A and v_B (for some $N = 2^n$) such that their Hamming distance is $N/2$ if they are not equal, and the answers have to be n-bit strings r_A and r_B.

Why is this game called pseudo-telepathy, and why is it interesting? If the parameter N is chosen large enough, this game cannot be won with certainty without communication. More precisely, the amount of communication required to win it is $\Omega(N)$. Hence winning this game proves the existence of telepathy... or of *quantum entanglement* [1]: The game *can* be won with certainty if the players can, in a preparation phase, not only agree on arbitrary classical, but also quantum information, more specifically, if they can generate and share so-called EPR pairs, i.e., maximally entangled states. They have the property that when measured by the players, the outcomes are the more strongly correlated the closer the chosen measurement bases are (and perfectly correlated if the bases are equal).

The two described facts lead to a lower bound on the amount of classical communication required for simulating quantum entanglement [1]. On the other hand, a simple demonstration experiment can be designed to convince an audience of the existence of quantum entanglement. For this purpose, however, the game has to be analyzed in more detail than previously done. In particular, it should be known what the minimal value of N is for which the game *cannot* be won without communication nor entanglement. (For large N, the game can be won with high probability by answering the questions by random hash values thereof; this is why small N are more interesting.)

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We approach this problem by showing a close connection of pseudo-telepathy and graph coloring. Using this link, we prove that for $N = 8$, the game can be won without any communication; this contrasts previous beliefs.

In the rest of this abstract we give a short description of some of the results in the full paper [2].

II. ...and Graph Colorings

Let $G = (V, E)$ be an undirected graph. We denote by $\mathrm{PT}(G, n)$ the (generalization of the) pseudo-telepathy game where the vertices are the possible questions asked, and where two questions are compatible (i.e., can be asked to the two participants simultaneously) if and only if they are identical or connected by an edge. The answers to be given are arbitrary n-bit strings.

The crucial observation is that whether $\mathrm{PT}(G, n)$ can be won or not *only* depends on the chromatic number $\chi(G)$ of G (and on n).

Theorem 1 $\mathrm{PT}(G, n)$ *can be won with certainty if and only if* $n \geq \log_2 \chi(G)$.

Proof Sketch. We can assume that the players' strategies are deterministic and identical. A strategy to win the game is hence a mapping $c : V \to \{0, 1\}^n$ such that for all $(u, v) \in E$, $c(u) \neq c(v)$. Such a mapping is called 2^n-coloring, and it exists if and only if $2^n \geq \chi(G)$. □

For $N = 2^n$, let now G_N be the graph corresponding to the game as described above (i.e., $V = \{0, 1\}^N$, $(u, v) \in E \longleftrightarrow d_H(u, v) = N/2$). $\mathrm{PT}(G_N, n)$ can, according to Theorem 1, be won if and only if $\chi(G_N) \leq N$. Since the codewords of a dual Hamming code of length $N - 1$ lead to a (maximal) clique in G_N, we have $\chi(G_N) \geq N$ for all N. It has been conjectured that the pseudo-telepathy game cannot be won for $N = 8$; Theorem 2 shows that this is incorrect.

Theorem 2 $\chi(G_8) = 8$.

The proof uses the fact that G_8 has independent sets of size $|V(G_8)|/8 = 32$ and the symmetry of the graph.

In [2], the graph G_{16} was investigated based on results from combinatorics, and the results strongly suggest that $\chi(G_{16}) > 16$. Thus $N = 16$ is probably the best parameter choice for an experiment demonstrating the existence of quantum entanglement.

References

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