

Hidden variables and the two theorems of John Bell

N. David Mermin

Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853-2501

Although skeptical of the prohibitive power of no-hidden-variables theorems, John Bell was himself responsible for the two most important ones. I describe some recent versions of the lesser known of the two (familiar to experts as the “Kochen-Specker theorem”) which have transparently simple proofs. One of the new versions can be converted without additional analysis into a powerful form of the very much better known “Bell’s Theorem,” thereby clarifying the conceptual link between these two results of Bell.

CONTENTS

I. The Dream of Hidden Variables	803
II. Plausible Constraints on a Hidden-Variables Theory	804
III. Von Neumann’s Silly Assumption	805
IV. The Bell-Kochen-Specker Theorem	806
V. A Simpler Bell-KS Theorem in Four Dimensions	809
VI. A Simple and More Versatile Bell-KS Theorem in Eight Dimensions	810
VII. Is the Bell-KS Theorem Silly?	811
VIII. Locality Replaces Noncontextuality: Bell’s Theorem	812
IX. A Little About Bohm Theory	813
X. The Last Word	814
Acknowledgments	814
References	814

Like all authors of noncommissioned reviews he thinks that he can restate the position with such clarity and simplicity that all previous discussions will be eclipsed.

J. S. Bell, 1966

I. THE DREAM OF HIDDEN VARIABLES

It is a fundamental quantum doctrine that a measurement does not, in general, reveal a preexisting value of the measured property. On the contrary, the outcome of a measurement is brought into being by the act of measurement itself, a joint manifestation of the state of the probed system and the probing apparatus. Precisely how the particular result of an individual measurement is brought into being—Heisenberg’s “transition from the possible to the actual”—is inherently unknowable. Only the statistical distribution of many such encounters is a proper matter for scientific inquiry.

We have been told this so often that the eyes glaze over at the words, and half of you have probably stopped reading already. But is it really true? Or, more conservatively, is it really necessary? Does quantum mechanics, that powerful, practical, phenomenally accurate computational tool of physicist, chemist, biologist, and engineer, really demand this weak link between our knowledge and the objects of that knowledge? Setting aside the metaphysics that emerged from urgent debates and long walks in Copenhagen parks, can one point to anything in the modern quantum theory that forces on us such an act of intellectual renunciation? Or is it merely reverence for the Patriarchs that leads us to deny that a measurement reveals a value that was already there, prior to the measurement?

Well, you might say, it’s easy enough to deduce from quantum mechanics that in general the measurement ap-

paratus disturbs the system on which it acts. True, but so what? One can easily imagine a measurement messing up any number of things, while still revealing the value of a preexisting property. Ah, you might add, but the uncertainty principle prohibits the existence of joint values for certain important groups of physical properties. So taught the Patriarchs, but as deduced from within the quantum theory itself, the uncertainty principle only prohibits the possibility of preparing an ensemble of systems in which all those properties are sharply defined; like most of quantum mechanics, it scrupulously avoids making any statements whatever about individual members of that ensemble. But surely indeterminism, you might conclude, is built into the very bones of the modern quantum theory. Entirely beside the point! The question is whether properties of individual systems possess values prior to the measurement that reveals them; not whether there are laws enabling us to predict at an earlier time what those values will be.

What, in fact, can you say if called upon to refute a celebrated polymath who confidently declares that “Most theoretical physicists are guilty of . . . fail[ing] to distinguish between a measurable indeterminacy and the epistemic indeterminability of what is in reality determinate. The indeterminacy discovered by physical measurements of subatomic phenomena simply tells us that we cannot know the definite position and velocity of an electron at one instant of time. It does not tell us that the electron, at any instant of time, does not have a definite position and velocity. [Physicists] . . . convert what is not measurable by them into the unreal and the nonexistent” (Adler, 1992, p. 300).

Are we, then, arrogant and irrational in refusing to consider the possibility of an expanded description of the world, in which properties such as position and velocity do have simultaneous values, even though nature has conspired to prevent us from ascertaining them both at the same time? Efforts to construct such deeper levels of description, in which properties of individual systems do have preexisting values revealed by the act of measurement, are known as hidden-variables programs. A frequently offered analogy is that a successful hidden-variables theory would be to quantum mechanics as classical mechanics is to classical statistical mechanics (see, for example, A. Einstein, in Schilpp, 1949, p. 672): quantum mechanics would survive intact, but would be understood in terms of a deeper and more detailed picture of the world. Efforts, on the other hand, to put our notori-

ous refusal on a more solid foundation by demonstrating that a hidden-variables program necessarily requires outcomes for certain experiments that disagree with the data predicted by the quantum theory, are called no-hidden-variables theorems (or, vulgarly, “no-go theorems”).

In the absence of any detailed features of a hidden-variables program, quantum mechanics is incapable of demonstrating that the general dream is impossible.¹ If the program consists of nothing beyond the bald assertion that such values exist, then while quantum physicists may protest, the quantum theory is powerless to produce a case in which experimental data can refute that claim, precisely because the theory is mute on what goes on in individual systems. A hidden-variables theory has to make *some* assumptions about the character of those preexisting values if quantum theory is to have anything to attack.

John Bell proved two great no-hidden-variables theorems. The first, given in Bell, 1966, is not as well known to physicists as it is to philosophers, who call it the Kochen-Specker (or KS) theorem² because of a version of the same argument, apparently more to their taste, derived independently by S. Kochen and E. P. Specker, 1967. I shall refer to it as the Bell-KS theorem. The second theorem, “Bell’s Theorem,” is given in Bell, 1964,³ and is widely known not only among physicists, but also to philosophers, journalists, mystics, novelists, and poets.

One reason the Bell-KS theorem is the less celebrated of the two is that the assumptions made by the hidden-variables theories it prohibits can only be formulated within the formal structure of quantum mechanics. One cannot describe the Bell-KS theorem to a general audience, in terms of a collection of black-box *gedanken* experiments, the only role of quantum mechanics being to provide *gedanken* results, which all by themselves imply that at least one of those experiments could not have been revealing a preexisting outcome. Bell’s Theorem, however, can be cast in precisely such terms.⁴ Indeed the hidden-variables theories ruled out by Bell’s Theorem rest on assumptions that not only can be stated in entire-

ly nontechnical terms but are so compelling that the establishment of their falsity has been called, not frivolously, “the most profound discovery of science” (Stapp, 1977).

The comparative obscurity of the Bell-KS theorem may also derive in part from the fact that the assumptions on which it rests were severely and immediately criticized by Bell himself: “That so much follows from such apparently innocent assumptions leads us to question their innocence.” We shall return to his criticism in Sec. VII.

A less edifying reason for the greater fame of Bell’s Theorem among physicists is that its proof is utterly transparent, while proving the Bell-KS theorem entails a moderately elaborate exercise in geometry. Physicists are simply less willing than philosophers to suffer through a few pages of dreary analysis to prove something they never doubted in the first place. So although all physicists know about Bell’s Theorem, most look blank when you mention Kochen-Specker or Bell-KS. Now, however, these particular grounds for such ignorance have been removed. Within the past few years new versions of the Bell-KS theorem have been found (Mermin, 1990b) that are so simple that even those physicists who regard such efforts as pointless can grasp the argument with negligible waste of time and mental energy. Besides making the argument so easy that even impatient physicists can enjoy it, one of the new forms of the Bell-KS theorem can also be readily converted into the striking new version of Bell’s Theorem invented by Greenberger, Horne, and Zeilinger,⁵ thereby shedding a new light on the relation between these two results of Bell.

II. PLAUSIBLE CONSTRAINTS ON A HIDDEN-VARIABLES THEORY

I now specify more precisely the general features of a hidden-variables theory. Quantum mechanics deals with a set of observables A, B, C, \dots and a set of states $|\Psi\rangle, |\Phi\rangle, \dots$. If we are given a physical system described by a particular state, then quantum mechanics gives us the probability of getting a given result when measuring one of the observables. More generally, if we have a group of mutually commuting observables, quantum mechanics asserts that we can do an experiment that measures them simultaneously and gives us the joint distribution for the values of each of the observables in that mutually commuting set.

We wish to entertain the heretical view that the results of a measurement are not brought into being by the act of measurement itself. This heresy takes the state vector to describe an ensemble of systems and maintains that in

¹David Bohm (Bohm, 1952) has, in fact, provided a hidden-variables theory that, if nothing else, serves as a proof that an unqualified refutation is impossible. I will return to Bohm theory in Sec. IX, merely noting here that it does exactly what Mortimer Adler wants, while remaining in complete agreement with quantum mechanics in its predictions for the outcome of any experiment.

²As mathematics, both results are special cases of a more powerful analysis by A. M. Gleason, 1957.

³In spite of the earlier publication date, Bell’s Theorem was proved *after* Bell proved his 1966 theorem. The manuscript of Bell, 1966, languished unattended for over a year in a drawer in the editorial offices of *Reviews of Modern Physics*.

⁴Several such formulations of Bell’s Theorem are given in Mermin, 1990a.

⁵Greenberger *et al.*, 1989. I have given a concise version of the Greenberger-Horne-Zeilinger argument in Mermin, 1990c and 1990d. An expanded discussion of their original argument can be found in Greenberger *et al.*, 1990.

each individual member of that ensemble every observable does indeed have a definite value, which the measurement merely reveals when carried out on that particular individual system. The quantum-mechanical rules, applied to a given state, give the statistics obeyed by those definite values in the ensemble described by that state. The uncertainty principle is not a restriction on the ability of observables to possess values in individual systems, but a limitation on the kinds of ensembles of individual systems it is possible to prepare, stemming from the unavoidable disturbance the state-preparation procedure imposes on the system. If two observables fail to commute, then the uncertainty principle does not prohibit both from having definite values in an individual system. It merely insists that it is impossible to prepare an ensemble of systems in which the values of neither observable fluctuate from one individual system to another.

To this kind of talk the well-trained quantum mechanician says "Rubbish!" and gets back to serious business. But is it possible to offer a better rejoinder? Is it possible to demonstrate not only that the innocent view is at odds with the prevailing orthodoxy, but that it is, in fact, directly refuted by the quantum-mechanical formalism itself, without any appeal to an interpretation of that formalism? A no-hidden-variables theorem attempts to provide such a refutation. It is only an attempt because any such theorem must make some assumptions on the nature of the hidden variables it excludes, which a persistent heretic can always call into question. Here is what I hope you will agree is a plausible set of assumptions for a straightforward hidden-variables theory.⁶

Given an ensemble of identical physical systems all prepared in the state $|\Phi\rangle$ described by observables A, B, C, \dots such a theory should assign to each individual member of that ensemble a set of numerical values for each observable, $v(A), v(B), v(C), \dots$, so that if any observable or mutually commuting subset of observables is measured on that individual system the results of the measurement will be the corresponding values.⁷ The theory should provide a rule for every state $|\Phi\rangle$ telling us how to distribute those values over the members of the ensemble described by $|\Phi\rangle$ in such a way that the statistical distribution of outcomes, for any measurement quantum mechanics permits, agrees with the predictions of quantum mechanics.

⁶But in Sec. VII we will come back, with Bell, to criticize one of them, so look them over carefully! At this point I deliberately refrain from calling the elusive culprit to your attention. It is my hope that you will find the assumptions sufficiently harmless to be curious whether any hidden-variables theory meeting such apparently benign conditions can indeed be ruled out by hard-headed quantum-mechanical calculation, rather than merely being rejected because it is in bad taste.

⁷Whether, and in what way, those values depend on new parameters or degrees of freedom is a detail of the particular hidden-variables theory and plays no role in what follows, except for the two-dimensional example of Bell described below.

Some of the constraints quantum mechanics imposes on the values are independent of the state $|\Phi\rangle$ we are examining. In particular, quantum mechanics requires that the result of measuring an observable be an eigenvalue of the corresponding Hermitian operator. Therefore only the eigenvalues of A can be allowed as values $v(A)$. Quantum mechanics further requires that if A, B, C, \dots is a mutually commuting subset of the observables then the only allowed results of a simultaneous measurement of A, B, C, \dots are a set of simultaneous eigenvalues. This correspondingly restricts the set of values $v(A), v(B), v(C), \dots$ possessed by an individual system. In particular, since any functional identity

$$f(A, B, C, \dots) = 0 \quad (1)$$

satisfied by a mutually commuting set of observables is also satisfied by their simultaneous eigenvalues, it follows that if a set of mutually commuting observables satisfies a relation of the form (1) then the values assigned to them in an individual system must also be related by

$$f(v(A), v(B), v(C), \dots) = 0 \quad (2)$$

Remarkably, some no-hidden-variables theorems arrive at a counterexample by considering only Eqs. (1) and (2), without even needing to appeal to the further constraints on the values imposed by the statistical properties of a particular state. The Bell-KS theorem is such a result. Others, of which Bell's Theorem is the most important example, require the properties of a special state to construct counterexamples. We shall examine in Sec. VII why it might be necessary for the scope of the counterexample to be restricted in this way. But before we begin, let us first look at a famous false start.

III. VON NEUMANN'S SILLY ASSUMPTION

Many generations of graduate students who might have been tempted to try to construct hidden-variables theories were beaten into submission by the claim that von Neumann, 1932, had proved that it could not be done. A few years later (see Jammer, 1974, p. 273) Grete Hermann, 1935, pointed out a glaring deficiency in the argument, but she seems to have been entirely ignored. Everybody continued to cite the von Neumann proof. A third of a century passed before John Bell, 1966, rediscovered the fact that von Neumann's no-hidden-variables proof was based on an assumption that can only be described as silly⁸—so silly, in fact, that one is led to

⁸While giving a physics colloquium on these matters I was taken to task by an outraged member of the audience for using the adjective "silly" to characterize von Neumann's assumption. I subsequently discovered that, like many penetrating observations about quantum mechanics, this one was made emphatically by John Bell: "Yet the von Neumann proof, if you actually come to grips with it, falls apart in your hands! There is *nothing* to it. It's not just flawed, it's *silly!* When you translate [his assumptions] into terms of physical disposition, they're nonsense. You may quote me on that: The proof of von Neumann is not merely false but *foolish!*" (Interview in *Omni*, May, 1988, p. 88.)

wonder whether the proof was ever studied by either the students or those who appealed to it to rescue them from speculative adventures.

A particular consequence of Eqs. (1) and (2) is that if A and B commute then the value assigned to $C = A + B$ must satisfy

$$v(C) = v(A) + v(B), \tag{3}$$

as an expression of the identity $C - A - B = 0$. Von Neumann's silly assumption was to impose the condition (3) on a hidden-variables theory even when A and B do not commute. But when A and B do not commute they do not have simultaneous eigenvalues, they cannot be simultaneously measured, and there are absolutely no grounds for imposing such a requirement. Von Neumann was led to it because it holds in the mean: for any state $|\Phi\rangle$, quantum mechanics requires, whether or not A and B commute, that

$$\langle \Phi | A + B | \Phi \rangle = \langle \Phi | A | \Phi \rangle + \langle \Phi | B | \Phi \rangle. \tag{4}$$

But to require that $v(A + B) = v(A) + v(B)$ in each individual system of the ensemble is to ensure that a relation holds in the mean by imposing it case by case—a sufficient, but hardly a necessary condition. Silly!

That the results of quantum mechanics are incompatible with values satisfying this condition is easy to see even in the two-dimensional state space that describes a single spin $\frac{1}{2}$. Let $A = \sigma_x, B = \sigma_y$. The eigenvalues of the Pauli matrices are ± 1 , so the values $v(A)$ and $v(B)$ are each restricted to be ± 1 . Thus the only values $v(A) + v(B)$ can have are $-2, 0$, and 2 . But $A + B$ is just $\sqrt{2}$ times the component of σ along the direction bisecting the angle between the x and y axes. As a result its allowed values are $v(A + B) = \pm\sqrt{2}$. Therefore a hidden-variables theory of this simple system cannot satisfy Eq. (3). But there is no reason to insist that it should! Indeed, having exposed the silliness in the von Neumann argument, Bell went immediately on to construct a hidden-variables model for a single spin $\frac{1}{2}$ that satisfies all the nonsilly conditions specified above. I now give this construction, but include it only to emphasize the nontriviality of the impossibility proofs we shall then turn to. Readers not interested in the details of Bell's counterexample can skip to Sec. IV.

In a two-dimensional state space every state is an eigenstate of the component σ_n of the spin along some direction \mathbf{n} ⁹:

$$\sigma_n |\uparrow_n\rangle = |\uparrow_n\rangle, \tag{5}$$

and every observable has the form

$$A = a_0 + \mathbf{a} \cdot \boldsymbol{\sigma}, \tag{6}$$

where a_0 is a real scalar and \mathbf{a} , a real three-vector. A set of observables A, B, C, \dots is mutually commuting if and only if the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ are all parallel. The eigenvalues of A , and hence the allowed values $v(A)$, are restricted to the two numbers

$$v(A) = a_0 \pm a, \tag{7}$$

where a is the magnitude of the vector \mathbf{a} . The simultaneous eigenvalues of a set of mutually commuting observables are given by choosing one sign in Eq. (7) for those observables whose vectors point one way along their common direction, and the opposite sign for those whose vectors point the other way. Because each observable A takes on only two values, the distribution of those values in a given state is entirely determined by the mean of A , which is given by

$$\langle \uparrow_n | A | \uparrow_n \rangle = a_0 + \mathbf{a} \cdot \mathbf{n}. \tag{8}$$

A rule associating with each observable one of its eigenvalues will yield simultaneous eigenvalues for mutually commuting observables if it always specifies the opposite sign in Eq. (7) for commuting observables associated with oppositely directed vectors. We require, in addition, for each state $|\uparrow_n\rangle$, that the rule specify a distribution of those values yielding the statistics demanded by Eq. (8). Here is a rule that does everything.¹⁰ Given a particular individual system from an ensemble described by the state $|\uparrow_n\rangle$, pick at random a second unit vector \mathbf{m} (which plays the role of the hidden variable) and assign to each observable A the values

$$\begin{aligned} v_n(A) &= a_0 + a, & \text{if } (\mathbf{m} + \mathbf{n}) \cdot \mathbf{a} > 0, \\ v_n(A) &= a_0 - a, & \text{if } (\mathbf{m} + \mathbf{n}) \cdot \mathbf{a} < 0. \end{aligned} \tag{9}$$

An elementary integration confirms that the mean over a uniform distribution of directions of \mathbf{m} of the value (9) of any observable in the state $|\uparrow_n\rangle$ is indeed given by the quantum-mechanical result (8):

$$\int \frac{d\Omega_m}{4\pi} v_n(A) = a_0 + \mathbf{a} \cdot \mathbf{n}. \tag{10}$$

IV. THE BELL-KOCHEN-SPECKER THEOREM

Having thus given an absurdly simple example of what had solemnly been declared impossible for the past three

⁹This is because every state can be related to $|\uparrow_z\rangle$ by a unitary transformation, but in a two-dimensional state space any unitary transformation, being a member of SU(2), represents a rotation.

¹⁰It is a little simpler than the one Bell gives. One can extend the rule to cover the case $(\mathbf{m} + \mathbf{n}) \cdot \mathbf{a} = 0$, but since this has zero statistical weight, I do not bother. Note that the values assigned to noncommuting observables do not satisfy von Neumann's additivity condition in individual members of the ensemble, although their average over the ensemble does, which is all quantum mechanics requires.

decades, Bell proceeded to show that the trick could no longer be accomplished in a state space of three or more¹¹ dimensions¹²; i.e., he gave a new no-hidden-variables proof that did not rely on the silly condition. I now give the full proof of this Bell-KS theorem, but here, too, I include it only to emphasize the much greater simplicity of the new versions that follow in Secs. V and VI, to which readers with no interest in the early history of the subject may jump without conceptual loss.

Just as it is convenient to use the algebra of spin $\frac{1}{2}$ to describe a two-dimensional state space, it is also convenient to describe the three-dimensional state space in terms of observables built out of angular momentum components for a particle of spin 1.¹³ The observables we consider are the squares of the components of the spin along various directions. Such observables have eigenvalues 1 or 0, since the unsquared spin components have eigenvalues 1, 0, or -1 . Furthermore the sums of the squared spin components along any three orthogonal directions u , v , and w satisfy

$$S_u^2 + S_v^2 + S_w^2 = s(s+1) = 2, \quad (11)$$

since we are dealing with a particle of spin 1 ($s=1$). Finally the squared components of the spin along any three orthogonal directions constitute a mutually commuting set.¹⁴

Suppose we are given a set of directions containing many different orthogonal triads, and the corresponding set of observables consisting of the squared spin components along each of the directions. Since the three observables associated with any orthogonal triad commute, they can be simultaneously measured, and the values such a measurement reveals for each of them, 0 or 1, must satisfy the same constraint (11) as the observables themselves. Thus two of the values must be 1 and the third, 0. We would have a no-hidden-variables theorem if we could find a quantum-mechanical state in which the statistics for the results of measuring any three observables associated with orthogonal triads could not be real-

ized by any distribution of assignments of 1 or 0 to every direction in the set, consistent with the constraint.

The Bell-KS theorem does substantially more than that: it produces a set of directions for which there is no way whatever to assign 1's and 0's to the directions consistent with the constraint (11), thereby rendering the statistical state-dependent part of the argument unnecessary. This is accomplished by solving the following problem in geometry: Find a set of three-dimensional vectors (i.e., directions) with the property that it is impossible to color each vector red (i.e., assign the value 1 to the squared spin component along that direction) or blue (i.e., assign the value 0) in such a way that every subset of three mutually orthogonal vectors contains just one blue and two red vectors.

The unpleasantly tedious part of the solution consists of showing that, if the angle between two vectors of different color is less than $\tan^{-1}(0.5) = 26.565$ degrees, then we can find additional vectors which, with the original two, constitute a set that cannot be colored according to the rules. Since all that matters is the direction of each vector, we can choose their magnitudes at our convenience. We take the blue vector to be a unit vector z defining the z axis and take the red vector a to lie in the y - z plane: $a = z + \alpha y$, $0 < \alpha < 0.5$.

We now make several elementary observations:

1. Since z is blue, x and y must both be red.¹⁵
2. Indeed, any vector in the x - y plane must be red, since one cannot have two orthogonal blue vectors. In particular $c = \beta x + y$ must be red, for arbitrary β . Particularly interesting values of β will be specified shortly.
3. Similarly, since a and x are red, any vector in their plane, and, in particular, $d = x/\beta - a/\alpha$ must be red.¹⁶
4. Because $a = z + \alpha y$, d is orthogonal to $c = \beta x + y$. Since both c and d are red, the normal to their plane must be blue, and therefore any vector in their plane, in particular, $e = c + d$ must be red.
5. But adding the explicit forms of c and d we see that $e = (\beta + \beta^{-1})x - z/\alpha$.
6. Since α is less than 0.5, $1/\alpha$ is greater than 2. Since $|\beta + \beta^{-1}|$ ranges between 2 and ∞ as β ranges through all real numbers, we can find a value of β such that e is along the direction of $f = x - z$. Changing the sign of β gives another e along the direction of $g = -x - z$.
7. Since e is red whatever the value of β , f and g must be red.
8. But f and g are orthogonal. The normal to their

¹¹His argument focuses on a space of exactly three dimensions, which can, however, be a subspace of a higher-dimensional space; the same remark applies to the new arguments in four and eight dimensions given in Secs. V and VI.

¹²Peculiar to two dimensions is the fact that all observables that commute with any nontrivial observable A necessarily commute with each other.

¹³Bell actually works with orthogonal projections, but the correspondence is entirely trivial: $S_x^2 = 1 - P_x$, etc. I find it more congenial to follow Kochen and Specker in using spin components, though the version of the argument I give is Bell's, not theirs.

¹⁴This is not a general property of angular momentum components but it does hold for spin 1, as is evident from the correspondence with orthogonal projections noted in the preceding footnote.

¹⁵As I mention each new vector, add it to the set.

¹⁶If you happen to be interested in counting how many vectors are in the uncolorable set we end up with, then whenever we add a red vector v in the plane of two orthogonal red vectors you should also add to the set, if they are not already present, a second red vector in that plane perpendicular to v , as well as a blue vector perpendicular to the plane.

plane is therefore blue and any vector in their plane is necessarily red.

9. But $z = -\frac{1}{2}f - \frac{1}{2}g$ is in the plane of f and g , and z is blue.

10. Contradiction! Therefore the set cannot be colored according to the rules if a and z have different colors.

The rest is genuinely trivial. We find an uncolorable set of directions by noting that, since 22.5 degrees $< \tan^{-1}(0.5)$, the z axis must have the same color as a direction 22.5 degrees away from it in the y - z plane, or we could produce an uncolorable set as described above. But that direction must then have the same color as the direction in the y - z plane another 22.5 degrees away from the z axis. Two more such steps gets us down to the y axis, which must thus have the same color as the z axis. Repeating this procedure in the y - x plane we conclude that the x axis must share the same color. But three mutually orthogonal axes cannot all have the same color: two must be red and one blue. Therefore the five directions in the y - z plane plus the four additional directions in the x - y plane plus the additional directions needed to carry out steps 1–10 above for each pair separated by 22.5 degrees constitute an uncolorable set.

Bell did not conclude his proof with these elementary remarks about steps of 22.5 degrees. Instead, after proving that differently colored directions must be more than a minimum angle apart, he simply noted that it was therefore impossible to associate a color with every direction, since any coloring of the sphere with just two colors obviously must have different colors arbitrarily close together. As a result, many philosophers characterize his proof as a “continuum proof” and prefer the argument Kochen and Specker independently gave a year later, which gives a slightly different (weaker) version of the minimum-angle theorem but explicitly displays a finite set of directions—117 of them—which cannot be colored according to the rules. Clearly the Bell argument as stated above also uses only a finite set of directions. But there is no point making a fuss about this because both arguments have now been superseded by an argument that is also state independent, whose algebraic part is even more elementary (appealing to no possibly unfamiliar result about the commutation of squares of orthogonal spin components), which requires no subsequent geometric analysis at all, and which uses far fewer observables.

The only price one pays for the simplicity is that the argument now requires a state space of at least four dimensions.¹⁷ So unless one has a special interest in prov-

¹⁷In hindsight this might have been guessed: if a no-hidden-variables theorem is impossible in two dimensions and rather complicated in three, extrapolation suggests that it might be easy in four.

ing no-hidden-variables theorems in three dimensions, one can safely declare the old Bell or Kochen-Specker versions of the theorem obsolete, sparing future generations of philosophers of science a painful rite of passage and making the result readily available even to physicists in ten minutes of an introductory quantum-mechanics course.

Before consigning the old argument to the history books, I digress to remark that the 117 directions seem to have held a great power over the philosophic imagination. Figure 1, for example, shows the cover illustration of a recent treatise on the philosophy of quantum mechanics (Redhead, 1987), emblazoned with the intricate diagram used by Kochen and Specker, 1967, to represent their set of uncolorable directions. Although the diagram is unfamiliar to all but a handful of quantum physicists, a distinguished philosopher of science regarded it as an appropriate icon for the entire subject.

Since 1967 other sets of uncolorable directions have been discovered with fewer vectors. The current world

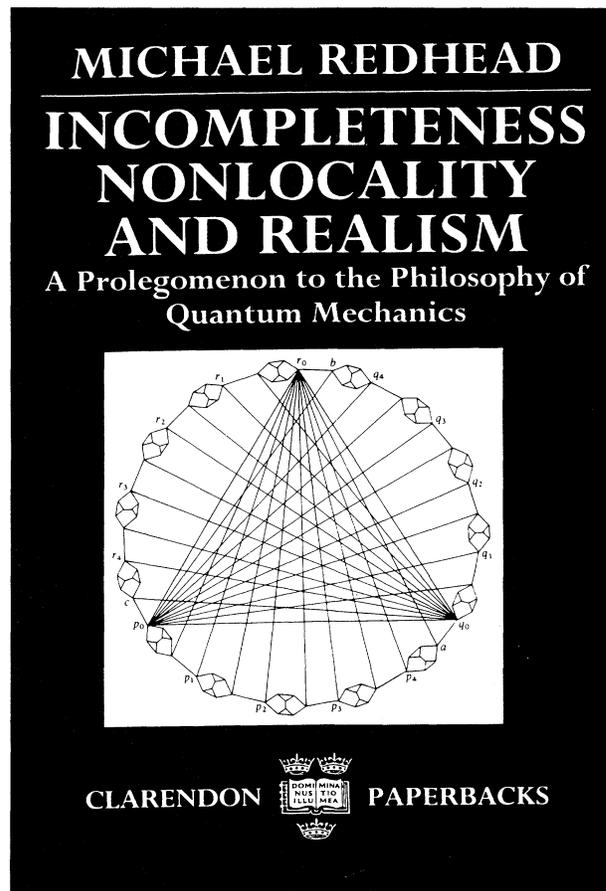
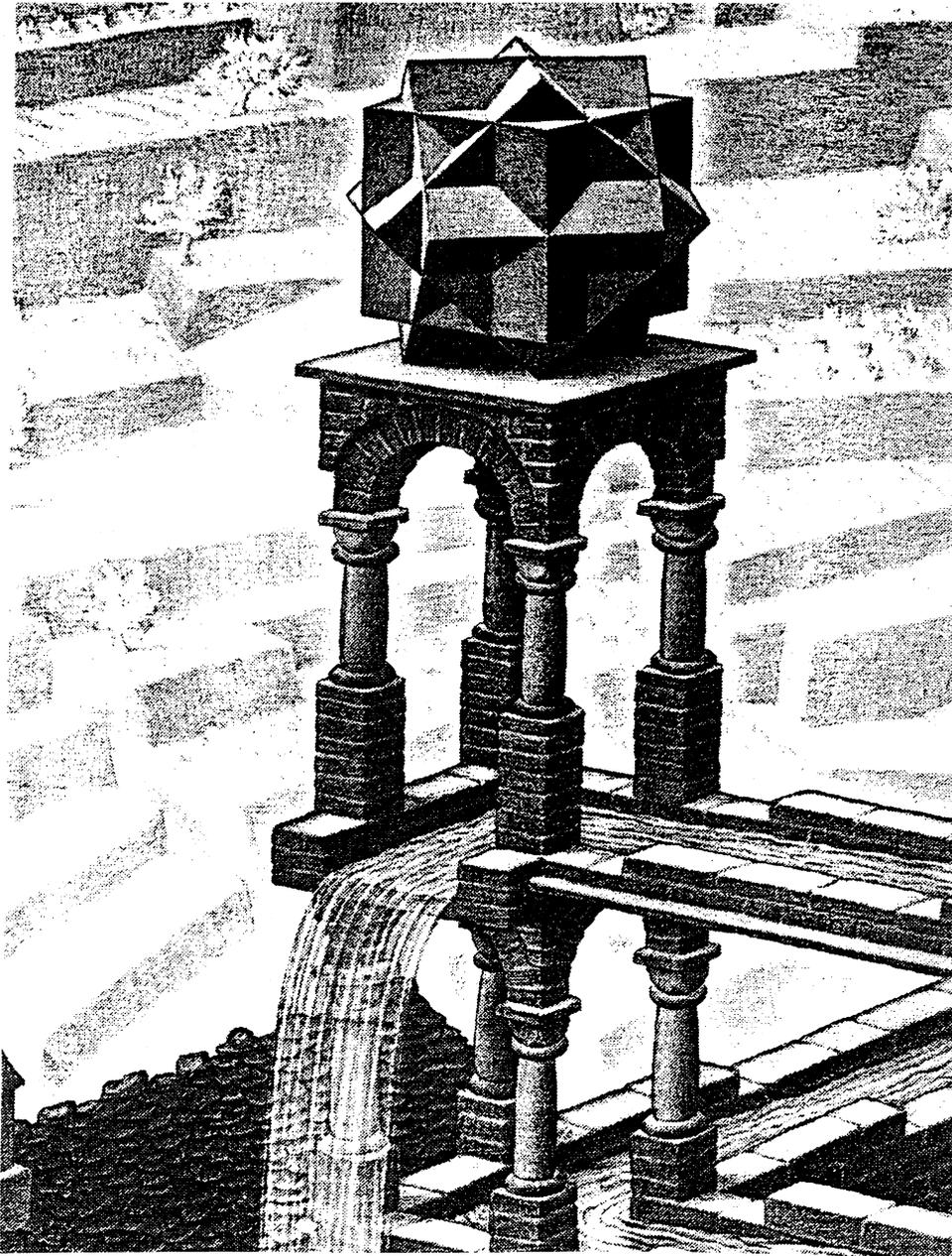


FIG. 1. The cover of Redhead, 1987, by permission of Oxford University Press. This reproduces the figure used in the proof of the Bell-KS theorem in Kochen and Specker, 1967. The figure contains 120 vertices representing 120 directions in three-space, but the pairs of directions a, p_0 , b, q_0 , and c, r_0 are the same, leaving the notorious set of 117 distinct directions.



©M. C. Escher / Cordon Art – Baarn – Holland.

FIG. 2. The tower on the left of M. C. Escher's engraving "Waterfall." © M. C. Escher/Cordon Art, Baarn, Holland. The ornament atop the tower consists of three superimposed cubes. One of the cubes has all its edges horizontal or vertical. The other two are given by rotating this one through 90 degrees about each of the two perpendicular horizontal lines that connect the midpoints of opposite vertical edges. The 33 uncolorable directions used in the proof of the Bell-KS theorem in Peres, 1991, lie along the lines connecting the common center of the cubes to their vertices and the centers of their edges and faces.

record holders are J. Conway and S. Kochen¹⁸ with 31, but Asher Peres, 1991, has found a prettier set of 33 with cubic symmetry, which can be exploited to give a proof of the no-coloring theorem that is more compact than Bell's. Roger Penrose has pointed out that Peres's set of 33 directions can be described as follows: take a cube and superimpose it with its 90-degree rotations about two perpendicular lines connecting its center to the midpoints of an edge. Peres's directions point to the vertices and to the centers of the faces and edges of the resulting set of

three interpenetrating cubes. This very figure occurs as a large ornament atop one of the two towers in M. Escher's famous drawing of the impossible waterfall, the relevant portion of which is shown in Fig. 2 (Escher, 1960).

V. A SIMPLER BELL-KS THEOREM IN FOUR DIMENSIONS

I now turn to the version of the Bell-KS theorem that works in a four-dimensional space.¹⁹ Our task is exactly

¹⁹This argument was inspired by an earlier version by A. Peres, 1990, that uses an even smaller number of observables, but applies only to an ensemble prepared in a particular state.

¹⁸S. Kochen, private communication.

the same as Bell, Kochen, and Specker faced in the three-dimensional case: we must exhibit a set of observables $A, B, C \dots$ for which we can prove that it is impossible to associate with each observable one of its eigenvalues, $v(A), v(B), v(C), \dots$ in such a way that all functional relationships between mutually commuting subsets of the observables are also satisfied by the associated values. The only difference is that now we can do the trick with many fewer observables and an entirely trivial proof.

In four dimensions it is convenient to represent observables in terms of the Pauli matrices for two independent spin- $\frac{1}{2}$ particles²⁰ σ_μ^1 and σ_ν^2 . The relevant properties of these observables are the familiar ones: the squares of each are unity, so the eigenvalues of each are ± 1 ; any component of σ_μ^1 commutes with any other component of σ_ν^2 ; when μ and ν specify orthogonal directions, σ_μ^i anticommutes with σ_ν^i for $i=1,2$; and $\sigma_x^i \sigma_y^i = i \sigma_z^i$ for $i=1,2$. Consider, then, the nine observables shown in Fig. 3, which it is convenient to arrange in groups of three on six intersecting lines that form a square. To prove that it is impossible to assign values to all nine observables we merely note that

(a) The observables in each of the three rows and each of the three columns are mutually commuting. This is immediately evident for the top two rows and first two columns from the left; it is true for the bottom row and right-hand column because in every case there is a pair of anticommutations.

(b) The product of the three observables in the column on the right is -1 . The product of the three observables in the other two columns and all three rows is $+1$.

(c) Since the values assigned to mutually commuting observables must obey any identities satisfied by the observables themselves, the identities (b) require the prod-

$$\begin{array}{ccc} \sigma_x^1 & \sigma_x^2 & \sigma_x^1 \sigma_x^2 \\ \sigma_y^2 & \sigma_y^1 & \sigma_y^1 \sigma_y^2 \\ \sigma_x^1 \sigma_y^2 & \sigma_x^2 \sigma_y^1 & \sigma_z^1 \sigma_z^2 \end{array}$$

FIG. 3. Nine observables leading to a very economical proof of the Bell-KS theorem in a state space of four or more dimensions. The observables are arranged in six groups of three, lying along three horizontal and three vertical lines. Each observable is associated with two such groups. The observables within each of the six groups are mutually commuting, and the product of the three observables in each of the six groups is $+1$ except for the vertical group on the right, where the product is -1 .

²⁰These are simply to be viewed as a convenient set of operators in terms of which to expand more general four-dimensional operators; we need not be talking about two spin- $\frac{1}{2}$ particles at all.

uct of the values assigned to the three observables in each row to be 1, and the product of the values assigned to the three observables in each column to be 1 for the first two columns and -1 for the column on the right.

But (c) is impossible to satisfy, since the row identities require the product of all nine values to be 1, while the column identities require it to be -1 .

I maintain that this is as simple a version of the Bell-KS theorem as one is ever likely to find²¹ and that it belongs in elementary texts on quantum mechanics as a direct demonstration, straight from the formalism, without any appeal to decrees by the Founders, that one cannot realize the naive ensemble interpretation of the theory on which the attempt to assign values is based. It is nevertheless susceptible to the same criticism that Bell himself immediately brought to bear against his own version of the theorem. Before turning to that criticism, however, I describe a comparably simple version of the Bell-KS theorem which works in an eight-dimensional state space²² that we shall find is capable of evading Bell's criticism in a way that the four-dimensional version is not. The eight-dimensional argument provides a direct link between the Bell-KS theorems and their illustrious companion, Bell's Theorem, when Bell's theorem is presented in the spectacular form recently discovered by Greenberger, Horne, and Zeilinger, 1989.

VI. A SIMPLE AND MORE VERSATILE BELL-KS THEOREM IN EIGHT DIMENSIONS

We construct our eight-dimensional observables out of three independent spins $\frac{1}{2}$, and consider the set of ten observables shown in Fig. 4, which it is now convenient to arrange in groups of 4 on five intersecting lines that form a five-pointed star. To prove that it is impossible to assign values to all ten observables note that

(a) The four observables on each of the five lines of the star are mutually commuting. This is immediately evident for all but the horizontal line, where it follows from the fact that interchanging the observables in each of the six possible pairs always requires a pair of anticommutations.

(b) The product of the four observables on every line of the star but the horizontal line is 1. The product of the

²¹Peres, 1991, recasts the argument as a no-coloring theorem for a set of 24 directions in four dimensions, thereby making it complicated again. The advantage of the four-dimensional argument over the traditional one in three dimensions is just that no such analysis is necessary.

²²That the three-spin form of the Greenberger-Horne-Zeilinger version of Bell's Theorem could be reinterpreted as a version of the Bell-KS theorem was brought to my attention by A. Stairs.

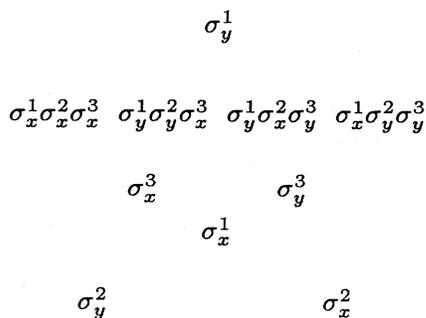


FIG. 4. Ten observables leading to a very economical proof of the Bell-KS theorem in a state space of eight or more dimensions. The observables are arranged in five groups of four, lying along the five legs of a five-pointed star. Each observable is associated with two such groups. The observables within each of the five groups are mutually commuting, and the product of the three observables in each of the six groups is +1 except for the group of four along the horizontal line of the star, where the product is -1.

four observables on the horizontal line is -1.

(c) Since the values assigned to mutually commuting observables must obey any identities satisfied by the observables themselves, the identities (b) require the product of the values assigned to the four observables on the horizontal line of the star to be -1, and the product of the values assigned to the four observables on each of the other lines to be +1.

Condition (c) requires the product over all five lines of the products of the values on each line to be -1. But this is impossible, for each observable is at the intersection of two lines. Its value appears twice in the product over all five lines, and that product must therefore be +1.

This hardly more elaborate eight-dimensional version of the theorem has an additional virtue that the four-dimensional version lacks. To see this and to see the connection with Bell's Theorem we turn, finally, to Bell's objection to his own argument.

VII. IS THE BELL-KS THEOREM SILLY?

In all these cases, as Bell pointed out immediately after proving the Bell-KS theorem, we have "tacitly assumed that the measurement of an observable must yield the same value independently of what other measurements must be made simultaneously." In Bell's three-dimensional example and in both the four- and eight-dimensional examples we required each observable to have a value in an individual system that would give the result of its measurement, *regardless of which of two sets of mutually commuting observables we chose to measure it with*. But since the additional observables in one of those sets do not all commute with the additional observables in the other, the two cases are incompatible. "These

different possibilities require different experimental arrangements; there is no *a priori* reason to believe that the results . . . should be the same. The result of an observation may reasonably depend not only on the state of the system (including hidden variables) but also on the complete disposition of the apparatus" (Bell, 1966).

This tacit assumption that a hidden-variables theory has to assign to an observable *A* the same value whether *A* is measured as part of the mutually commuting set *A, B, C, . . .* or a second mutually commuting set *A, L, M, . . .* even when some of the *L, M, . . .* fail to commute with some of the *B, C, . . .*, is called "noncontextuality" by the philosophers. Is noncontextuality, as Bell seemed to suggest, as silly a condition as von Neumann's—a foolish disregard of "the impossibility of any sharp distinction between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear," as Bohr²³ put it?

I would not characterize the assumption of noncontextuality as a silly constraint on a hidden-variables theory. It is surely an important fact that the impossibility of embedding quantum mechanics in a noncontextual hidden-variables theory rests not only on Bohr's doctrine of the inseparability of the objects and the measuring instruments, but also on a straightforward contradiction, independent of one's philosophic point of view, between some quantitative consequences of noncontextuality and the quantitative predictions of quantum mechanics.

Furthermore, there are features of quantum mechanics that seem strongly to hint at an underlying contextual hidden-variables theory as the only available explanation.²⁴ Most strikingly, although it is indisputable that measuring *A* with mutually commuting *B, C, . . .* requires a different experimental arrangement from measuring it with mutually commuting *L, M, . . .* whenever some of *L, M, . . .* fail to commute with some of *B, C, . . .*, it is nevertheless an elementary theorem of quantum mechanics that the joint distribution *p(a, b, c, . . .)* for the first experiment yields precisely the same marginal distribution *p(a)* as does the joint distribution *p(a, l, m, . . .)* for the second, in spite of the different experimental arrangements. If we do the experiment to measure *A* with *B, C, . . .* on an ensemble of systems prepared in the state Ψ and ignore the results of the other observables, we get *exactly the same* statistics for *A* as we would have obtained had we instead done the quite different experiment to measure *A* with *L, M, . . .* on that

²³N. Bohr in Schilpp, 1949 and cited in Bell, 1966. Bell's invocation of Bohr, to whom any hidden-variables theory would have been anathema, in order to dismiss the implications of his own no-hidden-variables theorem, thereby maintaining the viability of the hidden-variables program, was aptly characterized by Abner Shimony as "a judo-like maneuver."

²⁴An "only available explanation" is one to which the only alternative is the claim that no explanation is required.

same ensemble. The obvious way to account for this, particularly when entertaining the possibility of a hidden-variables theory, is to propose that both experiments reveal a set of values for A in the individual systems that is the same, regardless of which experiment we choose to extract them from. Putting it the other way around, a *contextual* hidden-variables account of this fact would be as mysteriously silent as the quantum theory on the question of why nature should conspire to arrange for the marginal distributions to be the same for the two different experimental arrangements.

Of course if the method of measuring A with mutually commuting B, C, \dots consists of successive filtrations—first measure A , then B , then C , etc.—and successive filtrations are also used to measure A with mutually commuting L, M, \dots , then if A is the first observable tested in either case, the resulting statistics for A alone will necessarily be the same in both cases, since we need not even decide which case to proceed with until after we have acquired the result of the A measurement. But this merely shifts the puzzle raised by the noncontextuality of quantum-mechanical probabilities to a new form: why should the statistical results of a sequential measurement of a set of mutually commuting observables be independent of the way we order them? Even more puzzling, why are those statistics unaffected if we change to quite a different way of determining them? We could, for example, measure three mutually commuting observables A , B , and C , each with eigenvalues 1 or 0 (like the squared spin components in the original Bell-KS argument) by measuring the single observable $4A + 2B + C$, the three-digit binary form of the result giving precisely the values of A , B , and C . If one is attempting a hidden-variables model at all, it seems not unreasonable to expect the model to provide the obvious explanation for this striking insensitivity of the distribution to changes in the experimental arrangement—namely, that the hidden variables are noncontextual.

There is, however, one class of no-hidden-variables theorems in which noncontextuality can be replaced by an even more compelling assumption, which brings us, finally, to Bell's Theorem (Bell, 1964).

VIII. LOCALITY REPLACES NONCONTEXTUALITY: BELL'S THEOREM

Suppose that the experiment that measures commuting observables A, B, C, \dots uses independent pieces of equipment far apart from one another, which separately register the values of A, B, C, \dots . And suppose that the experiment to measure A with the commuting observables L, M, \dots , not all of which commute with all of B, C, \dots , requires changes in the complete apparatus that amount only to replacing the parts that register the values of B, C, \dots with different pieces of equipment that register the values of L, M, \dots . And suppose that all these changes of equipment are made far away from the unchanged piece of apparatus that registers the value of

A . In the absence of action at a distance such changes in the complete disposition of the apparatus could hardly be expected to have an effect on the outcome of the A measurement on an individual system. In this case the problematic assumption of noncontextuality can be replaced by a straightforward assumption of locality.

Can we prove a Bell-KS theorem in which we assume noncontextuality only when it can be justified by locality? I know of no way to accomplish this trick that works for arbitrary states, but if one is willing to settle for a proof that works only for suitably prepared states, then it can easily be done. This was first accomplished in Bell's Theorem, which in its original form applies to a pair of far apart spin- $\frac{1}{2}$ particles *in the singlet state*. An analogous theorem can be established by a very minor modification of the eight-dimensional version of the Bell-KS theorem.²⁵ This new version of Bell's Theorem makes it clear that the use of a particular state is required to provide the information that is lost when one permits the assignment of noncontextual values only when noncontextuality is a consequence of locality.

To convert the eight-dimensional version of the Bell-KS theorem into a form of Bell's Theorem, we interpret the three vector operators σ^i , until now merely a convenient set from which to construct more general observables, as literally describing the spins of three different spin- $\frac{1}{2}$ particles, localized far away from one another. An examination of the ten observables appearing in Fig. 4 reveals that all but the four appearing on the horizontal line of the star describe spin components of a single isolated particle. Setting aside the four nonlocal observables, each of which is built out of the product of spin components of all three particles, we are left with six observables belonging to four sets, each containing three local observables, lying on the four nonhorizontal lines of the star. Each observable associated with a single particle appears in two of these sets, which differ in the selection of the pair of observables associated with the two faraway particles. For any of these six local observables, the assumption that the value assigned it should not depend on which pair of faraway components are measured with it is justified not by a possibly dubious assumption of noncontextuality, but by the condition of locality.

By dropping the noncontextual assignment of values to the four nonlocal observables, however, we break the chain of relations that led to a contradiction in the Bell-KS argument. We can rescue the argument by noting that because all four nonlocal observables *commute* with each other, they have simultaneous eigenstates. In an ensemble of individual systems prepared in such an eigenstate, the nonlocal observables all have definite values for valid and conventional quantum-mechanical reasons.

²⁵The modification converts it into the model of Greenberger, Horne, and Zeilinger, in the version I gave in Mermin, 1990c, 1990d.

These values play the same role in the new argument as the noncontextual values assigned them played in the earlier version, being related to the values of the appropriate sets of three local observables in exactly the same way.²⁶ The only difference is that because we now consider systems in an *eigenstate* of all four nonlocal observables, those four simultaneous values cannot fluctuate among the eight possible sets they might in general possess, but are fixed to a particular set of values. This further constraint does not alter the conclusion that there is no consistent way to assign values to all ten observables and thus no consistent assignment of values to the six local observables.

The eight-dimensional model of three spins $\frac{1}{2}$ therefore provides a conceptual link between the two theorems of John Bell that was not evident in their original forms. The difference between the two eight-dimensional arguments is that the Bell-KS version rules out the assignment of noncontextual values to arbitrary observables, while the Bell's Theorem version rules it out even when noncontextuality is restricted to cases in which it can be justified on the basis of locality. While both theorems demonstrate that the assignment is impossible, the demonstration based on locality is the more powerful result, since it applies even under a restricted use of noncontextuality.

Because the Bell-KS version applies to no-hidden-variables theories that are allowed to assign noncontextual values to a more general class of observables than in the Bell's Theorem version, the Bell-KS version does not need the properties of a particular state. In Bell's original versions of these theorems, where the arguments could not be set side by side, this appeared to be a compensating strength of the Bell-KS argument. In the new version, however, it is seen to be merely a technical consequence of the fact that by making a broader assignment of noncontextual hidden variables the Bell-KS argument can dispense with one of the stratagems the more powerful argument of Bell's Theorem requires to produce its counterexample.

It is instructive to see why we cannot convert the four-dimensional version of the Bell-KS theorem into an argument based on locality. In that argument (see Fig. 3) there are four local and five nonlocal observables that we now interpret as describing two far apart spin- $\frac{1}{2}$ particles. Each local observable can be measured with either of two other local observables that fail to commute with each other, associated with the other faraway particle. If we wish to make the assumption of noncontextuality only when it is required by the weaker assumption of locality, then we cannot assign noncontextual values to the five

nonlocal observables and need some other way to complete the chain leading to a contradiction. But in contrast to the eight-dimensional argument, the nonlocal observables do not all commute. It is thus no longer possible to assign values to all five by considering an ensemble of systems prepared in a simultaneous eigenstate. The theorem cannot be converted into a version of Bell's Theorem.

Note that locality can be used not only to justify the condition of noncontextuality but also to motivate further the attempt to assign values to the local observables in the first place. For in an ensemble of systems described by a simultaneous eigenstate of the nonlocal observables, the results of measuring any one of the local observables on an individual system can be determined prior to the measurement, by first measuring far away an appropriate set of two other local observables. Because the results of the measurements of the three local observables must be consistent with the eigenvalue of the observable that is their product, any two such results determine the third. As noted long ago by Einstein, Podolsky, and Rosen (Einstein *et al.*, 1935), in the absence of spooky actions at a distance it is hard to understand how this can happen unless the two earlier measurements are simply revealing properties of the subsequently measured particle that already exist prior to their measurement.

IX. A LITTLE ABOUT BOHM THEORY

Bell's favorite example of a hidden-variables theory, Bohm theory (Bohm, 1952), is not only explicitly contextual but explicitly and spectacularly nonlocal,²⁷ as it must be in view of the Bell-KS theorem and Bell's Theorem. In Bohm theory, which defies all the impossibility proofs, the hidden variables are simply the real configuration-space coordinates of real particles, guided in their motion by the wave function, which is viewed as a real field in configuration space. The wave function guides the particles like this²⁸: each particle obeys a first-order equation of motion specifying that its velocity is proportional to the gradient with respect to its position coordinates of the phase of the N -particle wave function, *evaluated at the instantaneous positions of all the other particles*. It is the italicized phrase which is responsible for the "hideous" nonlocality whenever the wave function is correlat-

²⁶For example, if σ_x^1 , σ_x^2 , and σ_x^3 are measured in an eigenstate of $\sigma_x^1\sigma_x^2\sigma_x^3$ with given eigenvalue, orthodox quantum mechanics requires the product of the three results to be equal to that eigenvalue.

²⁷This is noted in Bell, 1966, in which Bell raises the question of whether "any hidden-variables account of quantum mechanics *must* have this extraordinary character." (Remember, this was written before Bell, 1964!) Bell, 1982, reprinted as Chap. 17 of Bell, 1987, gives a more detailed discussion of Bohm theory from this perspective. Chapters 14 and 15 of Bell, 1987 give an exceptionally clear and concise exposition of Bohm theory.

²⁸I describe only spinless particles, but spin can also be handled.

ed.²⁹ One easily proves that if the wave function obeys Schrödinger's equation, then a distribution of initial coordinates of the particles given by $|\Psi_0|^2$ will evolve under these dynamics into $|\Psi_t|^2$ at time t .³⁰

If two particles are in a correlated state then, because the field guiding the second particle depends on the trajectory of the first, if a field is suddenly turned on in a region where the first particle happens to be, the subsequent motion of the second particle can be drastically altered in a manner that does not diminish with the distance between the two particles. Since measurements on each of a collection of noninteracting particles can be described by the action of just such fields, this gives non-contextuality with a vengeance.

X. THE LAST WORD

John Bell did not believe that either of his no-hidden-variables theorems excluded the possibility of a deeper level of description than quantum mechanics, any more than von Neumann's theorem does. He viewed them all as identifying conditions that such a description would have to meet. Von Neumann's theorem established only that a hidden-variables theory must assign values to non-commuting observables that do not obey in individual systems the additivity condition they satisfy in the mean—a result already evident from the trivial example of $\sigma_x + \sigma_y$. The Bell-KS theorems establish that in a hidden-variables theory the values assigned even to a set of mutually commuting observables must depend on the manner in which they are measured—a fact that Bohr could have told us long ago (although he would have disapproved of the whole undertaking). And Bell's Theorem establishes that the value assigned to an observable must depend on the complete experimental arrangement under which it is measured, even when two arrangements differ only far from the region in which the value is ascertained—a fact that Bohm theory exemplifies, and that is now understood to be an unavoidable feature of any hidden-variables theory.

To those for whom nonlocality is anathema, Bell's Theorem finally spells the death of the hidden-variables program.³¹ But not for Bell. None of the no-hidden-variables theorems persuaded him that hidden variables were impossible. What Bell's Theorem did suggest to

Bell was the need to reexamine our understanding of Lorentz invariance, as he argues in his delightful essay on how to teach special relativity (Bell, 1987, p. 12) and in Dennis Weaire's transcription of Bell's lecture on the Fitzgerald contraction (Bell, 1992). What is proved by impossibility proofs," Bell declared, "is lack of imagination."³²

ACKNOWLEDGMENTS

This work was supported by National Science Foundation under Grant No. PHY 9022796. This is a revised and expanded version of the text of the Bell Memorial Lecture given at the XIXth International Colloquium on Group-Theoretic Methods in Physics, Salamanca, July, 1992. (The earlier version is to appear in the proceedings of the Salamanca conference.) My treatment of these issues evolved through half a dozen general physics colloquia, given during the academic year 1991–1992, and has benefited from the thoughtful responses of skeptical members of those audiences. Many people contributed to my formulation and discussion of the new versions of both Bell theorems, with clever ideas, wise criticisms, or instructive failures to grasp points I foolishly thought I had made with transcendent clarity. I am especially indebted to Harvey Brown, Robert Clifton, Anthony Garrett, Kurt Gottfried, Daniel Greenberger, Jon Jarrett, Roger Penrose, Asher Peres, Abner Shimony, and Alan Stairs.

This essay is dedicated to the memory of my brother Joel Mermin (1939–1992), who loved to take long walks and simplify theorems.

REFERENCES

- Adler, Mortimer J., 1992, "Natural Theology, Chance, and God," in *The Great Ideas Today* (Encyclopedia Britannica, Chicago), pp. 288–301.
 Bell, J. S., 1964, "On the Einstein-Podolsky-Rosen Paradox," *Physics* **1**, 195–200.
 Bell, J. S., 1966, "On the problem of hidden variables in quantum mechanics," *Rev. Mod. Phys.* **38**, 447–452.
 Bell, J. S., 1982, "On the impossible pilot wave," *Found. Phys.* **12**, 989–999.
 Bell, J. S., 1987, *Speakable and Unsayable in Quantum*

²⁹If the wave function factors, then the phase is a sum of phases associated with the individual particles and the nonlocality goes away.

³⁰This is the way Bell presents Bohm theory. Bohm prefers to take another time derivative of the equation of motion for the particles to make it look more like $F = ma$, which he gets, with corrections to the classical force arising from what he calls the "quantum potential."

³¹Many people contend that Bell's Theorem demonstrates nonlocality independent of a hidden-variables program, but there is not general agreement about this.

³²Bell, 1982. Although I gladly give John Bell the last word, I will take the last footnote to insist that he is unreasonably dismissive of the importance of his own impossibility proofs. One could make a complementary criticism of much of contemporary theoretical physics: What is proved by possibility proofs is an excess of imagination. Either criticism undervalues the importance of defining limits to what speculative theories can or cannot be expected to accomplish.

- Mechanics* (Cambridge University, Cambridge).
- Bell, J. S., 1992, "George Francis Fitzgerald," *Phys. World* **5**, No. 9, 31–35. (Lecture at Trinity College, Dublin, 1989, as transcribed by Dennis Weaire.)
- Bohm, D., 1952, "A suggested interpretation of the quantum theory in terms of 'hidden variables,'" *Phys. Rev.* **85**, 66–179 (Part I); **85**, 180–193 (Part II).
- Einstein, A., B. Podolsky, and N. Rosen, 1935, "Can quantum-mechanical description of physical reality be considered complete?" *Phys. Rev.* **47**, 777–780.
- Escher, M., 1960, *The Graphic Work of M. C. Escher* (Hawthorn Books, New York), Plate 76, ornament atop the left tower.
- Gleason, A. M., 1957, "Measures on the closed subspaces of a Hilbert space," *J. Math. Mech.* **6**, 885–893.
- Greenberger, D. M., M. A. Horne, A. Shimony, and Z. Zeilinger, 1990, "Bell's theorem without inequalities," *Am. J. Phys.* **58**, 1131–1143.
- Greenberger, D. M., M. Horne, and A. Zeilinger, 1989, "Going beyond Bell's theorem," in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht), pp. 73–76.
- Hermann, G., 1935, "Die naturphilosophischen Grundlagen der Quantenmechanik (Anzug)," *Abhandlungen der Freis'schen Schule* **6**, 75–152.
- Jammer, M., 1974, *The Philosophy of Quantum Mechanics* (Wiley, New York), p. 273.
- Kochen, S., and E. P. Specker, 1967, "The problem of hidden variables in quantum mechanics," *J. Math. Mech.* **17**, 59–87.
- Mermin, N. D., 1990a, *Boojums All the Way Through* (Cambridge University, New York), Chaps. 10–12.
- Mermin, N. D., 1990b, "Simple unified form for the major no-hidden-variables theorems," *Phys. Rev. Lett.* **65**, 3373–3377.
- Mermin, N. D., 1990c, "What's wrong with these elements of reality?" *Phys. Today* **43**(6), 9.
- Mermin, N. D., 1990d, "Quantum mysteries revisited," *Am. J. Phys.* **58**, 731–734.
- Peres, A., 1990, "Incompatible results of quantum measurements," *Phys. Lett. A* **151**, 107–108.
- Peres, A., 1991, "Two Simple Proofs of the Kochen-Specker Theorem," *J. Phys. A* **24**, L175–L178.
- Redhead, M., 1987, *Incompleteness, Nonlocality, and Realism* (Clarendon, Oxford).
- Schilpp, P. A., Ed., 1949, *A. Einstein, Philosopher Scientist* (Library of Living Philosophers, Evanston, Ill).
- Stapp, H., 1977, *Nuovo Cimento* **40B**, 191 (1977).
- von Neumann, J., 1932, *Mathematische Grundlagen der Quanten-mechanik* (Springer-Berlin). English translation: *Mathematical Foundations of Quantum Mechanics* (Princeton University, Princeton, N.J., 1955).